MATH 195: Gödel, Escher, and Bach (Spring 2001)

Notes and Study Questions for Tuesday, March 20

Reading: Chapter VII–*Typographical Number Theory* (pp.204 – 213; to *Translation Puzzles*)
We'll also talk a bit about the previous dialogue, *Crab Canon,* left over from before break.

OK, this is it. Typographical Number Theory is the end of the road so far as formal systems is concerned. That might strike you as a bit disappointing, if you don't have more than a passing interest in number theory, but Gödel will show us that the implications go well beyond prime numbers. EVERY formal system, we will discover, can be mapped onto TNT. Furthermore, any problem with TNT is a problem with ANY formal system with equal or greater power as TNT.

By "problem" I mean:

a. Lack of consistency: Is it possible for TNT to prove mutually incompatible statements, like a statement and its negation?
b. Lack of completeness: Are there statements that are perfectly well formed within TNT but cannot be shown within the system to be either valid or invalid?

Fortunately, there isn't a LOT of difference between TNT and what you've already come across besides the concept of quantifiers. Unfortunately, this sometimes isn't an easy concept to grasp. And grasp it we must, because it is quantifiers that allows the system to make the kind of sweeping statements that we associate with grand truths.

STUDY QUESTIONS

***Crab Canon* [Repeated from notes for March 8]**

The dialogue is easy to read without much thought, but becomes more interesting when you pay more attention to it.

SQ1. Why is the dialogue called *Crab Canon*? Relate the structure of the dialogue to that of Bach's *Crab Canon* and / or Escher's *Crab Canon* (see Fig. 44 and listen to the file in Additional Material on the web).

SQ2. Are there places where the structural isomorphism is not exact? Why were those differences inserted?

Mysteriously interpolated into the dialogue is a brief explanation of the structure of DNA. Everyone is no doubt aware that DNA is a double-stranded molecule that contains all the genetic information within an organism. You might not realize that this information is contained twice -- once on each strand. Since the letters within DNA (G, A, T, C) pair with their counterparts (G with C, A with T), one strand is the inverse (or background) of the other. In the sequence shown in Fig. 43, one strand, read left to right, is identical to the other strand, read right to left. In the cell, the two strands are in fact read in opposite directions, and such crab canon-like structures are of immense importance in the function of DNA.

SQ3. Why do you think the structure breaks down midway (when Crab enters)?

SQ4. Why do you think Hofstadter has Crab say "TATA" at the end rather than "Goodbye"?

Chapter VIII *Typographical Number Theory*
What We Want to Be Able to Express in TNT

SQ5. The interpreted **PQ**-system makes statements about integer addition, the interpreted **TQ**-system makes statements about integer multiplication, and who knows what the **MIU**-system says. What does the interpreted **TNT**-system talk about?

Could you say "5 is a prime" in the propositional calculus? Sure: **P**. Could you say "6 is even" in the propositional calculus? Sure: **P**. There's something unsatisfying about this state of affairs. In TNT, the representation of these two statements are different from one another and captures the core of what the statements are trying to say.

SQ6. Satisfy yourself that each of the six more elemental representations (1') through (6') are indeed equivalent to the shorter representations (1) through (6). (5') is the trickiest.

Numerals; Variables and Terms; Atoms and Propositional Symbols

The atoms of the system should strike you as similar to those of the **PQ**-system. In the latter system, numbers were represented as a series of hyphens. In the **TNT**-system, numbers are represented as a series of **S**'s preceding **0**. These strings are interpretted as numbers. For example, **SSS0** is the number after the number after the number after 0, which is three.

You might think this is silliness. Clearly **SSS…S0** represents the number of **S**'s. Why not just write that number and be done with it, saving us the need to count all the letters? There are at least two reasons. First of all, in one more chapter it will be very convenient to have as few symbols as possible in TNT. Second of all, **S** is really an operation. It doesn't work merely on **0** but on any legitimate term of TNT. For example, **S(b+S0)** is perfectly OK and can be interpreted as "the number after b+1".

SQ7. Why was the letter **S** chosen?

SQ8. What numbers can be represented in the **TNT**-system? Do they differ from the numbers that can be represented in the **PQ**-system?

SQ9. We've encountered ***x***'s and ***y***'s before (e.g. "**<~*x*∧~*y*>** is equivalent to ~**<*x*∨*y*>**") as well as **P** and **P'**, etc. Now we have **a**, **b**, **a'**, etc. What's the difference?

SQ10. All the binary operations of the Propositional Calculus (**^**, **v**, **É**) are back. Is the syntax the same? (e.g., begins with an angle bracket, ends with an angle bracket) What, precisely, goes in place of **P**'s and **Q**'s?

Free Variables and Quantifiers

Now we come to something a good bit different. At the bottom of p.207 you see the statement:

(b + S0) = SS0

After interpreting this as "b+1 equals 2", you might be tempted to activate the part of your brain that does algebra. RESIST THE TEMPTATION! Leave the statement as "something plus one equals 2" and do not attempt to force truth on the statement by finding an appropriate value for "something". The statement remains neither true nor false until either (1) a value is substituted for **b** or (2) **b** is quantified.

"Quantifying" a variable means that the variable becomes governed by either ∃ or ∀. Take a look at how Hofstadter defines these symbols and then…

SQ11. Both quantifiers, the existential quantifier ∃ and the universal quantifier ∀, have several possible English equivalents. Supply as many equivalent English phrases for each quantifier as you can.

Don't be limited by "exists" and "all". Consider English sentences, like: *Any friend of yours is a friend of mine.* There are many more that can be translated as ∃ or ∀.

SQ12. What is the difference between the two statements in the middle of p.208:

$c:(c+S0) = SS0
" c:(c+S0) = SS0

Which, if either, strikes you as true?

SQ13. Hofstadter says near the bottom of p.208, "*Of course, no natural number has that property*". What property is he talking about and why is it that no natural number has it? Is the statement above that paragraph (beginning ~**$b:**…) true or false? Or neither?

SQ14. Is the statement shown below and found on p.209 true or false?

" c:(b + c) = (c + b)

Translating our Sample Sentences; **Tricks of the Trade**; and **Translation Puzzles**

The test of whether you understand the new syntax of TNT is to see whether you can grasp the connection between the English sentences on p.204 and their translations into TNT on pp.209-212 and even make the translations yourself.

SQ15. Hofstadter claims on p.210 that the two statements below are equivalent. Do they seem so to you?

~$b:(b · b) = SS0
"b:~(b · b) = SS0

SQ16. Hofstadter says he prefers the two translations of "1729 is the sum of two cubes" given near the bottom of page 210. Do you see why? (I don't. Mike does)

It is very easy to let your eyes glaze over as you go through the explanations of the sample sentences, particularly sentences 4 and 5. Don't do it! Take each little point one at a time and let the English sentence grow. Let's go through Sentence 4 together. It says, "No sum of two positive cubes is itself a cube". "Sum of two cubes…" I can translate that crudely into symbols (not TNT symbols!) as: $\mathbf{b^3 + c^3}$. The statement seems to assert that this sum isn't a cube. Sounds like we can never get numbers to fit into $\mathbf{a^3 = b^3 + c^3}$.

Is that true? Well, no. $0^3 = 0^3 + 0^3$. However, the original statement specified <u>positive</u> cubes: 0^3 doesn't count. So how do we represent positive numbers? Of course, the word "positive" is not part of TNT. The numbers that ARE part of TNT are **0**, **S0**, **SS0**, and so forth: all the positive natural numbers AND 0. Hoffstadter uses a trick to represent just positive numbers:

	1st	2nd	3rd	4th	5th
Legal numbers in TNT:	**0**	**S0**	**SS0**	**SSS0**	**SSSS0**
Positive numbers:	**S0**	**SS0**	**SSS0**	**SSSS0**	**SSSSS0**

Clearly, the first positive number is one more than the first legal number, the second positive number is one more than the second legal number, and so forth. Just by saying **Sb**, where **b** is any legal number (i.e. natural number), we capture the positive natural numbers.

OK, now let's see what Hoffstadter did with this sentence. He started more simply with a specific example, "No sum of two positive cubes is 7" which I'll translate crudely to "There are no positive natural numbers **b** and **c** that satisfy $\mathbf{7 = b^3 + c^3}$. HE translated that into (middle of p.211):

~∃b:∃c:SSSSSSS0 = (((Sb · Sb) · Sb) + ((Sc · Sc) · Sc))

This may look imposing, certainly more imposing than my crude translation, but how different is it? Let me translate it back: **SSSSSSS0** is interpreted as 7. **Sb**, as we have seen, is "some positive natural number". **Sc** is another positive natural number, which may or may not be the same as **Sb**. Plugging these interpretations back in and we get something pretty reasonable.

However, the original statement is supposed cover not 7 but for all cubes, so Hofstadter replaced **SSSSSSS0** with a representation of $\mathbf{a^3}$ and quantified it with **∀**, giving:

∀a:	**~∃b:∃c:**	**((a · a) · a)**	**=**	**(((Sb · Sb) · Sb)**	**+**	**((Sc · Sc) · Sc))**
For all **a**	There are no natural numbers **b** and **c** such that	a^3 cube	equals	$(b+1)^3$ (positive number)^3	plus	$(c+1)^3$ (positive number)^3

SQ17. Does the alternate representation of sentence 5 (beginning **~$a:$b:$c:**) seem equivalent to you?

SQ18. Try the translation puzzles on pp.212-213.

Application of Quantifiers to Verbal Logic

The universe of TNT is natural numbers, but "all" and "there exists" are also useful concepts with regard to fruit, the stock market, and a variety of other arenas. Just as TNT specified the atoms its rules act upon, so must one have in mind the atoms on which quantifiers act upon in English sentences. The sentence "*Everyone has a mother*" sounds reasonable, so long as it is restricted to people and excludes certain marine algae. You might render this:

"person: person has mother

SQ19. With that in mind, try your hand at translating the following sentences into partially symbolic sentences that use one or more quantifiers.

a. All apples are red
b. Some apples are red
c. No apples are red
d. No one loves everybody
e. Everyone loves someone
f. Everybody loves somebody sometime
(I don't know how to translate this one, but it makes a great song)