# MATH 195: Gödel, Escher, and Bach (Spring 2000)
## Notes and Study Questions for Tuesday, February 27

**Reading:** *Chromatic Fantasy, And Feud* (pp.177-180)
  Chapter VII: *The Propositional Calculus* (pp.178-188, to *Rounding out the. . .*)

The dialogue should not pose any problems for you. It's meant to make a simple point, dealt with in Chapter VII, and I think you'll get it. . . right between the eyes. The Chapter is another story. Here is one more chapter devoted to a formal system, but finally we're given one with a good deal of power to express complex thoughts. The simpler formal systems of the previous chapters can be seen as practice enabling us to absorb the prepositional calculus. In the NEXT chapter (Chapter VIII: *Typographical Number Theory*), a modification of the formal system you learn here will give us all we need to follow Gödel in his proof.

Much of what you read should be familiar: the symbols of a formal system, as opposed to variables in the metalanguage used to describe the system; the rules of a formal system; well-formed strings; the line-by-line derivations. The difference is in scale. The prepositional calculus has a lot more symbols and a lot more rules. What it does *not* have is axioms. It is in the nature of this system that *we* are asked to provide temporary axioms and see what comes of that. What we do with this system is ask, "*If* **x** *were an axiom, then what would come of that?*"

There is also the difference in the scale of complexity. Interpretations of the formal statements often seem beyond comprehension. They're not. Just as a magician works his craft by waving his arms and spreading a fog of mystery, so do these statements demoralize you by <u>looking</u> complicated. Your defense is to take them apart and go through their derivations patiently, step by step.

## Study Questions

### Chromatic Fantasy, And Feud

The title of this dialogue is derived from Bach's *Chromatic Fantasy and Fugue*. I don't know of any deep connection between that piece and this dialogue, apart from their titles. You can judge for yourself by listening to the piece (see Additional Material on the web).

1. What do you think is Hofstadter's point in including this dialogue?

### The Propositional Calculus

Here's a new formal system. As always, write down the *symbols*, the *axioms*, and the *rules*. It's more difficult this time than previously, because these elements are thrown piecemeal at us (so we're not overwhelmed, I suppose). So keep a page devoted to these elements of the system, to be filled in with the elements as you encounter them.

**Well-Formed Strings**

2. Hofstadter introduces a new term, "atoms". Are they part of the system (like "**U**" in the **MIU**-system) or variables within a metalanguage (like "*x*" in the **MIU**-system)? Can we apply "atoms" to the **MIU**- and **pq**-systems? If so, then what are their atoms?

3. What's so recursive about the four Formation Rules?

4. Go through the nine examples of well-formed strings (middle of p.182) . Can you catch a misprint?

5. Does the nine lines constitute a derivation? Is the last line consequently a theorem?

6. Hofstatder claims that you can check the well-formedness of a string by a top-down procedure that runs the Rules of Formation backwards. How would you go about doing this? Use the last of the nine well-formed strings as an example. [HINT: Those angle brackets, "<" and ">" can greatly simplify the procedure]

7. Which of the eight strings at the bottom of page 182 are well-formed?

**More Rules of Inference** and **The Fantasy Rule**

8. What's the difference between "Formation Rules" and "Rules of Inference"? Isn't the Rule of Separation just Formation Rule #2 run backwards?

9. What do you think "^" and "~" stand for? What does "stand for" mean?

10. Hofstadter devotes more than a page to talking about the Fantasy Rule without really stating it. <u>You</u> state it, in as few words as possible. [HINT: the string **<<P^Q>É<Q^P>>** is not <u>the</u> Fantasy Rule but rather the result of one application of the rule]

     This section introduces two new symbols: "[" and "]". They're different from the others in the system in that they cannot appear in theorems. They're more procedural, indicating that a derivation is pushing into a fantasy or popping out of one.

11. In derivations using the **pq**-system, each line is a theorem (plus a brief explanation). Which lines in the derivation in the middle of p.184 are theorems of the Propositional Calculus? Why?

12. What is the intended interpretation of the symbol "**É**"?

**Recursion and the Fantasy Rule** and **The Converse of the Fantasy Rule**

13. What does "carry over" mean?

14. The Carry-over Rule refers only to theorems one level higher. What about theorems *two* levels higher? Can they be carried over?

15. Do you understand how the derivation in the middle of p.185 works?

16. Why is the section entitled "*The Converse of the Fantasy Rule*" given its title? <u>Is</u> the Rule of Detachment the converse of the Fantasy Rule? Why?

### The Intended Interpretation of the Symbols

For the previous formal systems, interpretations were sometimes not given at all (**MIU**-system) or given as an afterthought (**pq**-system). In the Propositional Calculus, the interpretation is of paramount importance, as the intent of the system is to capture the essence of strict logical thought. However, while our thinking is usually directed to producing results that make a difference in our lives, the rules of the Propositional Calculus care nothing about profundity. Their role is merely to *preserve* truth: *If* some statement is true, *then* these other truths follow.

17. Expand the following theorem:

$$\textbf{<PÉ<QÉ<P^Q>>>}$$

using the following interpretations:

**P** represents "*The stock market will rise tomorrow*"

**Q** represents "*I can free up a lot of cash*"

18. Translate the derivation in the middle of p.185 into a coherent English paragraph, using the above interpretations of **P** and **Q** (or make up your own).

19. Given the intended interpretations of the symbols of the Propositional Calculus, do the Rules of Inference make sense? Consider in particular the rules that describe how certain strings are interchangeable.