Math 195: Gödel, Escher, and Bach (Spring 2001)

Welcomes you to Exam #3

RULES OF THE GAME: Same as always.

- It's open book, open notes, closed people, until the exam is handed back. Answers go on an ANSWER SHEET, not the exam.
- EXCEPTION: Turn in the last page of the exam, with the blanks filled in.
- Thoughts go on a THOUGHTS SHEET.
- CLUES are available if needed.

POTENTIALLY HELPFUL SUGGESTION:

There is a wide range of abilities of those taking this exam, and it is virtually impossible to write an exam that is not too easy for some without being too difficult (and too long) for others. Don't get hung up by what you can't do. Focus on what you <u>can</u> do. It's better to put down what thoughts you have concerning a question, even though you recognize that you don't have a satisfactory answer.

The Questions

1. (1) If you have neither received nor given aid regarding this exam, nor have you gained or given knowledge concerning a previous or future administration of this exam, then sign your name. Otherwise sign someone else's name.

2. (1) On the first exam, many of you indicated that you thought in-class discussions in small groups were valuable for you in understanding how to do the problems. Do you still think so? What do you feel is the best use of our time in class? If you like you can also answer anonymously on the web (Discussion Board).

3. (1) The web allows us to post potentially helpful items as needed and to plan material for the next class based on what we saw in the previous class. On the other hand, many people don't avail themselves of material or discussion on the web, and there may be the cost of inconvenience by the web-based approach. Should we switch instead to a paper-based course with handouts?

4. (3) Is the string **<SS0·b ∨ ~SS0·b>** well formed? How can it be translated?

- **A.** Well-formed: "All the world's a stage"
- **B.** Well-formed: "That which we call a rose, by any other name would smell as sweet.
- **C.** Well-formed: "A horse! A horse! My kingdom for a horse!"
- **D.** None of the above (supply your own answer)

5. (8) Translate the following into symbols to the degree possible. In the interest of partial credit, you may wish to include several translation steps, becoming more 'symbolic' on each line (a separate line for each step would be helpful to us).

- **a.** Hang gliding isn't for everyone
- **b.** To every thing there is a season, and a time to every purpose under heaven

6. (10) Derive from the propositional calculus the notion that anything follows logically if you presume a falsehood. If you like, you can derive the symbolic representation of the following: Presuming that pigs can't fly, then if pigs COULD fly, I'd be a millionaire.

7. (10) A common way to prove a logical proposition is to assume that the proposition is false and show that this assumption leads to a contradiction.

a. Choose one of the strings below that you think adequately represents this strategy (only one does) and show by verbally parsing (analyzing) the string why you think so.

A. < < < P v Q > ^ ~P > É Q >
B. < < < P É Q > ^ < Q É R > > É < P É R > >
C. < < < ~P É Q > ^ < ~P É ~Q > > É P >

b. Construct a truth table that demonstrates that the string you chose is indeed valid.

8. (6) True or False:

a. It is possible to have a formal system without any axioms.
b. It is possible to have a formal system without any criteria for well-formedness.
c. It is possible to have a well-formed formula of TNT that does not contain the symbol “=”.

9. (10) Say whether each statement below is true or false. If true, give some value of **c** and some value of **d** for which the statement holds. If false, then give some value of **c** and value of **d** for which the statement fails.

a. " c: $d: (SSS0 · c) = d

b. $d: " c: (SSS0 · c) = d

10. (8) Complete each statement and <u>give the name of a rule illustrated by the completed statement</u>.

Example: *If roses are red and violets are blue, then it follows that…*
Answer: *…roses are not red or violets are not blue. (via De Morgan's rule)*

a. If every good boy deserves fudge[1] and Johann is a good boy, then…

b. To establish that NOT all cows eat grass, it would suffice to exhibit that…

11. (8) Convert each of the following statements in English to an equivalent statement in English (same truth value) as indicated. <u>Justify your answer by citing a rule of Propositional Calculus</u>.

a. Either stones are soft OR the sky is blue. (convert to a legal IF … THEN statement)

b. Either stones are soft OR the sky is blue. (convert to a legal AND statement)

[1] Note that the first letters of Every Good Boy Deserves Fudge correspond to the lines of the treble clef. Therefore, no one can say that this exam is absolutely devoid of musical content.

12. (16) We wish to show that (any number) times 1 is equal to (the same number you started with). In symbols, this can be written $\forall c: (c\cdot S0) = c$. The proof will be by induction. You are to help out by filling in the blanks. Note the division of the proof into the usual sections.

SECTION I: Preliminaries to derivation of $< X\{c\} \supset X\{Sc|c\} >$)

(1)	$\forall a:\forall b: (a + Sb) = S(a + b)$	Axiom 3
(2)	$\forall a: (a + Sc) = S(a + c)$	Specification, from line (1), replace **b** by **c**
(3)	$((c\cdot 0) + Sc) = S((c\cdot 0) + c)$	Specification, from line (2), replace **a** by **(c·0)**
(4)	$S((c\cdot 0) + c) = ((c\cdot 0) + Sc)$	Symmetry, from line (3)
(5)	$\forall a:\forall b: (a\cdot Sb) = ((a\cdot b) + a)$	Axiom 5
(6)	$\forall a: (a\cdot S0) = ((a\cdot 0) + a)$	Specification, from line (5), replace **b** by **0**
(7)	$(c\cdot S0) = ((c\cdot 0) + c)$	Specification, from line (6), replace **a** by **c**
(8)	$((c\cdot 0) + c) = (c\cdot S0)$	________________________________
(9)	$\forall a:\forall b: (a\cdot Sb) = ((a\cdot b) + a)$	Axiom 5
(10)	$\forall a: (a\cdot S0) = ((a\cdot 0) + a)$	________________________________
(11)	$(Sc\cdot S0) = ((c\cdot 0) + Sc)$	Specification, from line (10), replace **a** by **Sc**
(12)	$((c\cdot 0) + Sc) = (Sc\cdot S0)$	Symmetry, from line (11)

SECTION II: Derivation of $< X\{c\} \supset X\{Sc|c\} >$)

(13)	[	Push
(14)	$(c\cdot S0) = c$	premise (i.e., X{c})
(15)	$((c\cdot 0) + c) = (c\cdot S0)$	Carry-over rule, from line (8)
(16)	_______________	Transitivity, lines (14) and (15)
(17)	$S((c\cdot 0) + c) = Sc$	________________________________
(18)	$S((c\cdot 0) + c) = ((c\cdot 0) + Sc)$	Carry-over rule, from line __________
(19)	$((c\cdot 0) + Sc) = (Sc\cdot S0)$	Carry-over rule, from line (12)
(20)	$S((c\cdot 0) + c) = (Sc\cdot S0)$	____________, from lines (18) and (19)
(21)	_______________	Symmetry, from line (20)
(22)	$(Sc\cdot S0) =$ ________	Transitivity, from lines _____ and (21) (i.e. X{Sc|c})
(23)	]	Push
(24)	$< (c\cdot S0) = c \supset (Sc\cdot S0) = Sc >$	Fantasy rule (i.e., $< X\{c\} \supset X\{Sc\|c\} >$)
(25)	_______________	Generalization, from line (24)

SECTION III: Derivation of $X\{0|c\}$

(26)	$\forall a: (a\cdot 0) = 0$	Axiom 4
(27)	$(a\cdot 0) = 0$	Specification, from line (26), replace **a** with **a**
(28)	$\forall a:\forall b: (a\cdot b) = (b\cdot a)$	derivation omitted[2]
(29)	$\forall b: (a\cdot b) = (b\cdot a)$	Specification from line (28), replace **a** with **a**
(30)	_______________	Specification from line (29), replace _______
(31)	$(0\cdot a) = (a\cdot 0)$	Symmetry, from line ________
(32)	$(0\cdot a) =$ __________	Transitivity, from lines (31) and _______
(33)	$\forall a: (0\cdot a) = 0$	Generalization, from line (32)
(34)	$(0\cdot S0) = 0$	Specification, from line (33), replace **a** by **S0** (i.e., X{0|c})

SECTION V: Bottom line

(35)	$\forall c: (c\cdot S0) = c$	Induction, from lines _____ and (34)

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[2] We would need many tens more lines here (similar to long derivation on pp.225-227).