Exam 1 Solutions

1. (1) If you have neither received nor given aid regarding this exam, nor have you gained or given knowledge concerning a previous or future administration of this exam, then sign your name. Otherwise sign someone else's name.

SOLUTIONS: There was a lot of variability on this question. We gave credit for any reasonable answer.

2. (2) What has been most useful to you in wading through the intricacies of the course? What would you like to see more of?

YOUR RESPONSES: Stay tuned. We'll let you know what you said after we digest the information.

3. (4) How many quizzes did you take of the five made available? (If you don't remember, don't worry about it)

4. (10) Did you hand in the assigned homework taken from Problem Set 2?

5. (4) Make up a strange loop, either in language or some other medium. If your loop duplicates someone else's the points will be divided between you, so think up something no one else will.

SOLUTIONS: If we find the time we may post some of the student responses. There were many examples of 1-step loops, some 2-step loops, and a few loops of more than 2 steps.

6. (10) Welcome to *Let's Make a Logical Deal!* There are three curtains, **A, B,** and **C**. Behind one curtain is Your Fantasy Fulfilled. Behind the other two are cans of tuna fish. You *hate* tuna fish. To increase your chances (heh, heh) I will give you what's behind TWO of the curtains if you make a true statement, but make a false statement and you get NOTHING!

6a. What do you say to make your dreams come true?

6b. Why do you say it? Provide either a series of short statements (one per line) illustrating your reasoning or a table showing that your response will be effective. No paragraphs please!

[*We anticipate that many of you might have difficulties with this question and have prepared hints to offer you if you need them*]

SOLUTION: We thought of two alternative statements, either

"You will either give me My Fantasy Fulfilled or you will not give me tuna fish"

or "You will not give me both cans of tuna".

We will present the reasoning in two possible formats (you only needed one). We think these formats help to enhance the clarity of the argument. I'll use the second statement, "You will not give me both cans of tuna" to illustrate.

1. If the statement “You will not give me both cans of tuna” were false, then …
    a. according to the rules of the game, you would give me nothing
    b. according to the statement, you would give me both cans of tuna
    c. these two outcomes are not compatible with each other
    d. therefore the statement I made can’t be false.

2. If the statement “You will not give me both cans of tuna” were true, then …
    a. according to the rules of the game, I would receive two prizes
    b. according to the statement, the two prizes wouldn’t both be tuna fish
    c. therefore my prizes include My Fantasy Fulfilled.

This format is nice for proving that your statement solves the puzzle, though it may not be quite as helpful in thinking up the statement in the first place.

ALTERNATIVELY, you might use the table below. Incompatible situations are excluded (gray), and what remains is possible. The table gives you: (a) insight into what kind of statement you want (incompatible with two options, compatible with the third), and (b) a test for whether a candidate statement works. If you don't feel inspired, one strategy is to go through every simple statement you can think of, and when they're exhausted, go through compound statements.

	Possible outcomes		
	I get NOTHING	I get 2 TUNAs	I get MY FANTASY and 1 TUNA
A true statement	incompatible	compatible	compatible
A false statement	compatible	incompatible	incompatible
You will give me only one can of tuna	false	false	true
You will not give me tuna fish	true	false	false
. . .			
You will give me tuna or my fantasy	false	true	true
You will give me my fantasy or nothing	true	false	true
You will not give me two cans of tuna fish	true	false	true

The table may be more helpful to you in dreaming up a statement to make: you can see that you want to make a true statement AND you want a statement that prevents you from receiving two cans of tuna fish.

7. (6) State whether each of the strings below is or is not a theorem of the **MIU**-system. If it is, then prove it.

7a. **MUIIU**, using **MI** as the sole axiom

SOLUTION: (notice that if you “work backwards”, converting each U to three I’s you’ll have eight I’s in all --- parentheses are used as visual aids below)

MI goes to **MII** via Rule 2
MII goes to **MIIII** via Rule 2
MIIII goes to **MIIIIIIII** via Rule 2
M(III)IIIII goes to **MUIIIII** via Rule 3

MUII(III) goes to **MUIIU** via Rule 3

7b. **MIUI**, using **MI** as the sole axiom

SOLUTION: (this one's trickier, mostly because you need to figure out when to use Rule 1)

MI goes to **MII** via Rule 2
MII goes to **MIIII** via Rule 2
MIIII goes to **MIIIIIIII** via Rule 2
MIIIIIIII goes to **MIIIIIIIIU** via Rule 1
MIIIIII(III)U goes to **MIIIIIIUU** via Rule 3
MIIIIIIUU goes to **MIIIIII** via Rule 4
MI(III)I goes to **MIUI** via Rule 3

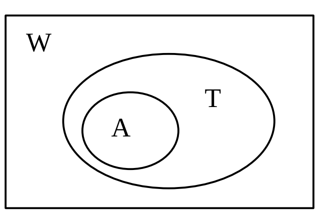
8. (6) For each of the strings below, state whether, within the **pq-**system, it is:

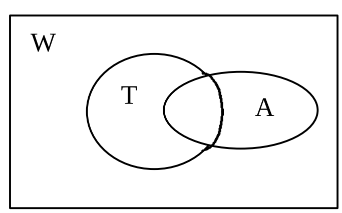
(**A**) An axiom
(**T**) A theorem and not an axiom
(**W**) A well-formed string but not a theorem or an axiom
(**N**) A string that is not well-formed

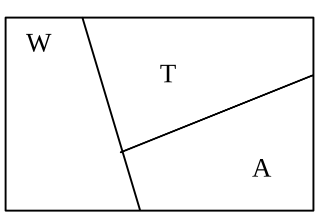
SOLUTIONS IN CAPS:

8a. - p - - - q - - - - **(T) A THEOREM BUT NOT AN AXIOM**
8b. - - - - p q - - - - **(N) A STRING THAT IS NOT WELL-FORMED**
8c. - - - p - - q - - - **(W) A WELL-FORMED STRING, NOT A THEOREM**

9. (6) Which of the following responses best captures the relationships between the axioms, the theorems, and the well-formed strings of a formal system. Sets are labeled as in problem #8 with A for axioms, T for theorems, and W for well-formed strings.

A.


B.


C.


D. It depends on the system.

SOLUTION: Diagram A is correct since

1. An axiom is a theorem (a theorem that is given), so set A is contained in set T.

2. A theorem has to be well-formed, so set T is contained in set W.

10. (12) A new formal system, the *EXAMI-system*, has the following description:

The symbols of the system are the letters **E, X, A, M, I**

The rules of the system are

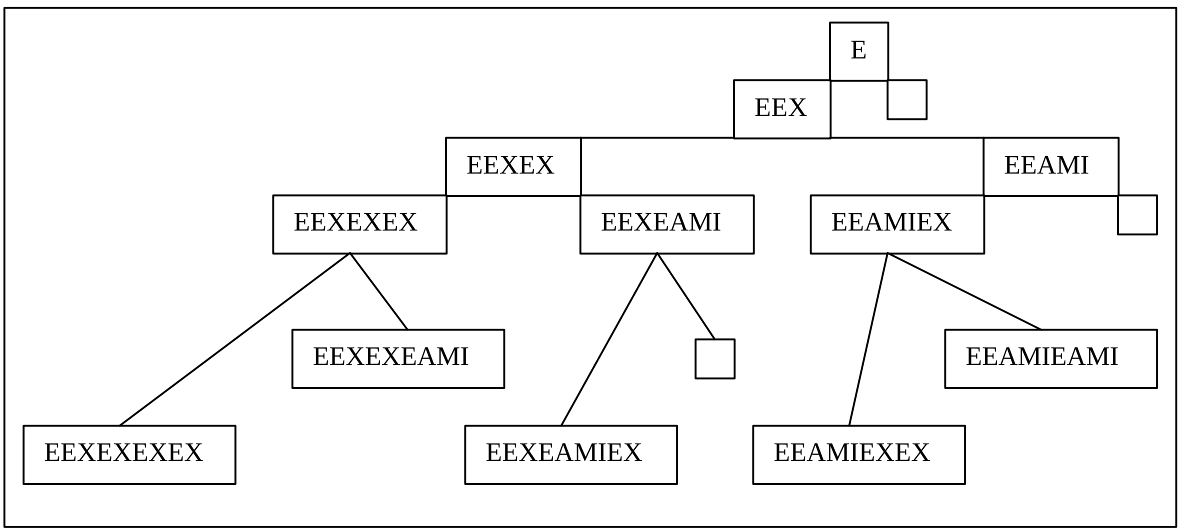
RULE 1: if *y* is a theorem, then *y***EX** is also a theorem
RULE 2: if *y***X** is a theorem, then *y***AMI** is also a theorem

The sole axiom of the system is

AXIOM 1: **E**

10a. Draw 5 levels of the generating tree for this system

SOLUTION: Rule 1 branches to the left and, where applicable, Rule 2 branches to the right.



10b. State two metatheorems for this system. Your goal is to describe large collections of theorems in more-or-less plain English. You are encouraged to use formats such as: *"All strings of the form _______ are theorems"* or *"Every theorem in this system _______"* .

SOLUTIONS: There are several possibilities including

"*All strings of the form EXEXEXEX… are theorems (repeated blocks of EX)*"

"*All strings of the form EAMIEAMIEAMIEAMI… are theorems (repeated blocks of EAMI)*"

"*Every theorem in this system begins with E*"

"*Every theorem in this system that is not an axiom ends in either X or I*"

"*Every theorem in this system contains E followed by blocks of EX or EAMI*"

"*Every theorem in this system contains an odd number of symbols*"

11. (16) Let's finally solve the **MU**-puzzle, through a top-down approach. In class we expressed the rules of the **MIU** system (p34-35) via the shorthand

RULE 1:	***x*I**	goes to	***x*IU**
RULE 2:	**M*x***	goes to	**M*xx***
RULE 3:	***x*III*y***	goes to	***x*U*y***
RULE 4:	***x*UU*y***	goes to	***xy***

As part of problem 6 of Problem Set 3, you figured out the strings that could possibly precede MU in a derivation. Your solution was

MU	to be proved
MIII	leads to **MU** via Rule 3
MUUU	leads to **MU** via Rule 4

Take a moment to pause and let your mind remember…

11a. Continue this list by writing every string that can immediately precede **MIII**, leading to **MIII**, through one of the rules of production. You can say to yourself, "What string could produce **MIII** through one application of Rule #1?" If there is such a string, write it down (followed by "Rule #1"), then go on to Rule #2, etc.

SOLUTION: **MIII** could be immediately preceded by **MIIIUU** using Rule 4, by **MIIUUI** using Rule 4, by **MIUUII** using Rule 4, or by **MUUIII** using Rule 4.

Rule 1 won't help because **MIII** does not end in **U**.

Rule 2 won't help because **MIII** does not have an even number of symbols following **M**.

Rule 3 won't help because **MIII** does not contain a **U**.

We've continued this process even further and produced the following strings guaranteed to lead in some number of steps to **MU** (the list is not exhaustive):

MIIIIIIU	MIIUIIII	MIUUIIUU	MUIIIUUU	MUUIUUII
MIIIIIUI	MIIUUI	MIUUIUUI	MUIIUUIU	MUUUIIIU
MIIIIUII	MIIUUIUU	MIUUUUII	MUIUUIIU	MUUUUIII
MIIIUIII	MIIUUUUI	MUIII	MUUIII	MUUUUU
MIIIUU	MIUIIIII	MUIIIIII	MUUIIIUU	
MIIIUUUU	MIUUII	MUIIIU	MUUIIUUI	

11b. Go through the list of strings and note how many **I**'s appear in each string, then jump into I-mode and see if you can find a general rule to describe the number of **I**'s that can appear in strings generated by this procedure. What is the rule you found?

SOLUTION: It seems that the number of **I**'s is always a multiple of 3. Note that the strings given have only 0, 3, or 6 **I**'s, but clearly these aren't the only possibilities. From Rule 2 alone, it is clear that higher numbers of **I**'s are readily obtained.

11c. State your rule as a metatheorem about the **MIU**-system. State the metatheorem in the following way:

*<u>Metatheorem</u>: All strings capable of leading to **MU** through the rules of the **MIU**-system have [restate the property you found in **11b**].*

SOLUTION: several possibilities including

*… have 0, 3, 6, 9, 12, … **I**'s,* *… have a number of **I**'s that is a multiple of 3,*

*… have 3*k **I**'s, where k = 0,1,2,3,4,….* Paradox: this hardest one is easiest to work in part d.

11d. Prove that metatheorem in the following way:

*Suppose that **Mx** is a string of the **MIU**-system that contains n **I**'s [substitute for n the rule you found in **11c**]. Then:*

· *If a string exists that could produces **Mx** through Rule #1, that string would have [fill in] **I**'s. (Demonstrate through the definition of the rule)*

· *[Make similar statements with respect to Rules #2, 3, and 4].*

SOLUTION:

Suppose that **Mx** is a string of the **MIU**-system that contains *3*k* **I**'s . Then:

- *If a string exists that produces **Mx** through Rule #1, that string would have 3*k **I**'s.* (Demonstrate through the definition of the rule) I'll argue instead of demonstrating --- all Rule 1 does is to add a **U,** so the number of **I**'s is unchanged.
- *If a string exists that produces **Mx** through Rule #2, that string would have 3*k/2 **I**'s.* This is because Rule 2 doubles the portion of the string that follows the **M** (pursuing this further, note that k would have had to have been even .... Anyway, the important point is that the number of ***I***'s in the preceding string is a multiple of 3). Try it -- if some number is doubled and produces a multiple of three, then the original number is also a multiple of three.

  If you wanted something more certain, then consider this:

  - By hypothesis, ***x*** is a multiple of three. That means that ***x*** divided by three gives some integer.
  - Rule #2 works by doubling the number of hyphens, so ***x*** is also a multiple of two. That means that ***x*** divided by two also gives some integer
  - Since 2 and 3 don't share a common factor, ***x*** divided by three and then by two gives some integer. Call that integer ***n***. So ***x*** = 3*2****n***.
  - The string that produced **M*x*** has half the number of hypens, i.e. 3*2****n***/2 = 3****n***. So that string is also a multiple of three.
- *If a string exists that produces **Mx** through Rule #3, that string would have 3*(k-1) **I**'s.* Applying Rule 3 replaces three **I**'s by a single **U**. Reducing by three a number that is a multiple of three gives you a number that's STILL a multiple of three.
- *If a string exists that produces **Mx** through Rule #4, that string would have 3*k **I**'s.* Eliminating **UU** will not change the number of **I**'s in the string.

Note that it isn't enough to show for some particular string that these relationships hold. You need to show it in general.

11e. Even if you can't prove the metatheorem in **11d**, you can still use it to try to shed light on whether **MU** can be produced from **MI** within the **MIU**-system. In doing so, you will need to relate how **MU** and **MI** fit into the metatheorem. Relate your thoughts in short phrases (one per line) put in a logical order, not paragraphs.

SOLUTION:

The *Metatheorem* of **11c** says that every string that could possibly lead to **MU** contains a number of **I**'s that's a multiple of 3.

Since the number of **I**'s in **MI** is not a multiple of 3, the implication of the *Metatheorem* of **11c** is that the string **MI** cannot produce **MU** through any sequence of applications of the four given rules.

You could have written the above in short statements, as follows:

- Every string that leads to **MU** has a number of **I**'s that's a multiple of 3 (*Metatheorem* of **11c**)
- **MI** has only one **I**, not a multiple of 3.
- Therefore, **MI** cannot be a string that can lead to **MU.**

Very simple. All the work was in proving the metatheorem. Metatheorems are wonderful.

Many of you considered the possibility that **MI** might somehow produce a string with multiple of 3 **I**'s, but that misses the strength of the metatheorem. If the metatheorem is true, then **MI** (nor any other string without a multiple of 3 **I**'s) can't possibly lead to **MU** by any number of steps.

Many of you also went off in a different direction, arguing that **MI** can be lengthened only by powers of two, which can never lead to a multiple of three. This isn't quite right. **MIIIIIIIIII** (10 **I**'s) can readily be produced from **MI**, and 10 is not a power of two. This approach can modified to lead to a proof that **MU** cannot be derived from **MU**, but it's a lot more work than using the metatheorem of **11c**.