

Math 195: Gödel, Escher, and Bach (Spring 2001)

Welcomes you to Exam #1

Most people consider exams as times to be EXAMined, to be put to the TEST, to have their innards squeezed out so that they're open to INSPECTION and EVALUATION. Actually, to our mind what's more important is what goes IN during this period, not what comes out. An exam can be a time of concerted effort when insights finally click into place.

To that end, we encourage you to go beyond what you think you know, and when you've reached your limit, to come to one of us for help – yes, help – during the exam. We may exact a small price for this help (a reduction in points for the question) so that you view help as a distinct second choice to going as far as you can on your own. But the last choice should be giving up on a question.

RULES OF THE GAME: This is an open book exam. It is an open notes exam. Needless to say, it is not an open people exam. This applies particularly to communication between people taking the exam at different times.

ANSWER SHEET: Put answers on any paper you like, but NOT on the exam itself (it would just be a mess). Don't bother rewriting the question on your answer sheet, just the answers themselves – as neatly and logically as you can. The exam is yours to keep.

THOUGHT SHEET: Reserve your answer sheet for answers. However, you are encouraged to write on a separate sheet (labeled "thoughts") the thought processes that led you to those answers or, if you're stuck, what sticks you.

WEIGHTS OF QUESTIONS: See the numbers in the parentheses for each question.

NEED A CLUE? If you've exhausted your own resources with respect to one of the questions, consider coming to Mike or Jeff for a hint to get you back on track. Some hints may cost a fraction of the total points you receive for the question, but that may be better than getting nothing at all. You'll be advised of any charge before receiving a hint.

GEB News

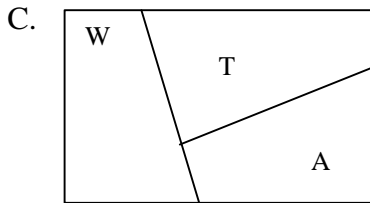
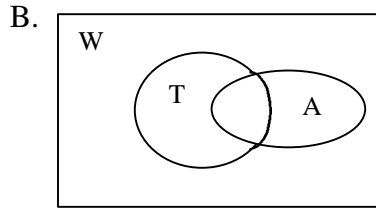
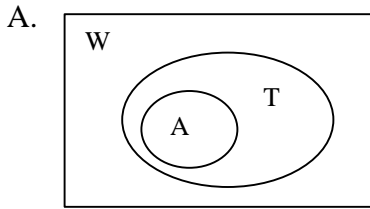
- Just a reminder, there IS class on Tuesday, with the usual reading, notes and quiz. Don't relax TOO much after the exam
- Special opportunity for the creatively inclined. The reading for Tuesday includes the one-sided conversation entitled *Sonata for Unaccompanied Achilles*. Consider writing the OTHER side of the conversation. If you choose to do this, and I hope you do, send Jeff (JElhai@Richmond.Edu) a copy of your work before class on Tuesday.

The Questions

1. (1) If you have neither received nor given aid regarding this exam, nor have you gained or given knowledge concerning a previous or future administration of this exam, then sign your name. Otherwise sign someone else's name.
2. (2) What has been most useful to you in wading through the intricacies of the course? What would you like to see more of?
3. (4) How many quizzes did you take of the five made available? (If you don't remember, don't worry about it)
4. (10) Did you hand in the assigned homework taken from Problem Set 2?
5. (4) Make up a strange loop, either in language or some other medium. If your loop duplicates someone else's the points will be divided between you, so think up something no one else will.
6. (10) Welcome to *Let's Make a Logical Deal!* There are three curtains, **A**, **B**, and **C**. Behind one curtain is Your Fantasy Fulfilled. Behind the other two are cans of tuna fish. You *hate* tuna fish. To increase your chances (heh, heh) I will give you what's behind TWO of the curtains if you make a true statement, but make a false statement and you get NOTHING!
 - 6a. What do you say to make your dreams come true?
 - 6b. Why do you say it? Provide either a series of short statements (one per line) illustrating your reasoning or a table showing that your response will be effective. No paragraphs please!

[We anticipate that many of you might have difficulties with this question and have prepared hints to offer you if you need them]
7. (6) State whether each of the strings below is or is not a theorem of the **MIU**-system. If it is, then prove it.
 - 7a. **MUIIU**, using **MI** as the sole axiom
 - 7b. **MIUI**, using **MI** as the sole axiom
8. (6) For each of the strings below, state whether, within the **pq**-system, it is:
 - (**A**) An axiom
 - (**T**) A theorem and not an axiom
 - (**W**) A well-formed string but not a theorem or an axiom
 - (**N**) A string that is not well-formed
 - 8a. - p - - - q - - - -
 - 8b. - - - - p q - - - -
 - 8c. - - - - p - - q - - - -

9. (6) Which of the following responses best captures the relationships between the axioms, the theorems, and the well-formed strings of a formal system. Sets are labeled as in problem #8 with A for axioms, T for theorems, and W for well-formed strings.



D. It depends on the system.

10. (12) A new formal system, the *EXAMI-system*, has the following description:

The symbols of the system are the letters **E, X, A, M, I**

The rules of the system are

RULE 1: if y is a theorem, then $y\mathbf{EX}$ is also a theorem

RULE 2: if $y\mathbf{X}$ is a theorem, then $y\mathbf{AMI}$ is also a theorem

The sole axiom of the system is

AXIOM 1: **E**

10a. Draw 5 levels of the generating tree for this system (AXIOM 1 by itself counts as level one of the tree).

10b. State two metatheorems for this system. Your goal is to describe large collections of theorems in more-or-less plain English. You are encouraged to use formats such as:

“All strings of the form _____ are theorems”

or “Every theorem in this system _____” .

11. (16) Let's finally solve the **MU**-puzzle, through a top-down approach. In class we expressed the rules of the **MIU** system (p34-35) via the shorthand

- RULE 1: xI goes to xIU
 RULE 2: Mx goes to Mxx
 RULE 3: $xIIIy$ goes to xUy
 RULE 4: $xUUy$ goes to xy

As part of problem 6 of Problem Set 3, you figured out the strings that could possibly precede **MU** in a derivation. Your solution was

- MU** to be proved
MIII leads to **MU** via Rule 3
MUUU leads to **MU** via Rule 4

Take a moment to pause and let your mind remember...

11a. Continue this list by writing every string that can immediately precede **MIII**, leading to **MIII**, through one of the rules of production. You can say to yourself, "What string could produce **MIII** through one application of Rule #1?" If there is such a string, write it down (followed by "Rule #1"), then go on to Rule #2, etc.

We've continued this process even further and produced the following strings guaranteed to lead in some number of steps to **MU** (the list is not exhaustive):

MIIIIIIU	MIIUIIII	MIUUIIUU	MUIIIUUU	MUUIUUUI
MIIIIUII	MIIUUI	MIUUUIUI	MUIIUUIU	MUUUIIIU
MIIIIUII	MIIUUIIU	MIUUUIII	MUIUUIIU	MUUUIIII
MIIUIIII	MIIUUUII	MUIII	MUIIII	MUUUIIU
MIIUIIU	MIUIIIII	MUIIIIII	MUIIIIIU	
MIIUIUUU	MIUIII	MUIIIU	MUIIIUIU	

11b. Go through the list of strings and note how many **I**'s appear in each string, then jump into I-mode and see if you can find a general rule to describe the number of **I**'s that can appear in strings generated by this procedure. What is the rule you found?

11c. State your rule as a metatheorem about the **MIU**-system. State the metatheorem in the following way:

*Metatheorem: All strings capable of leading to **MU** through the rules of the **MIU**-system have [restate the property you found in 11b].*

11d. Prove that metatheorem in the following way:

Suppose that Mx is a string of the **MIU**-system that contains n **I**'s [substitute for n the rule you found in 11f]. Then:

- If a string exists that could produce Mx through Rule #1, that string would have [fill in] **I**'s. (Demonstrate through the definition of the rule)
- [Make similar statements with respect to Rules #2, 3, and 4].

11e. Even if you can't prove the metatheorem in 11d, you can still use it to try to shed light on whether **MU** can be produced from **MI** within the **MIU**-system. In doing so, you will need to relate how **MU** and **MI** fit into the metatheorem. Relate your thoughts in short phrases (one per line) put in a logical order, not paragraphs.