Fractionally Coloring the Plane

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Joint with Landon Rabern
Slides available on my webpage

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Coloring the Plane

Goal:

Color the plane so points at distance 1 get distinct colors.

- vertices are points of $\mathbb{R}^2$
- two vertices adjacent if points are at distance 1

Unit distance graph is any subgraph of this graph.

Min number of colors needed is $\chi(\mathbb{R}^2)$.

What's known:

- $\chi(\mathbb{R}^2) \geq 4$
- (a) The Moser spindle
- (b) The Golomb graph
Coloring the Plane

**Goal:** Color the plane so points at distance 1 get distinct colors.
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---

What's known?

3

1

2

(a) The Moser spindle

3

2

2

3

2

(b) The Golomb graph

So $\chi(\mathbb{R}^2) \geq 4$
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What's known?
- $3^2$?
- $1^3$?
- $2^2$?
- $3^2$?
- $2^2$?
- $1^3$?
- $1^3$?

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**What’s known?**

![Diagram](image)  
(a) The Moser spindle
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So $\chi(\mathbb{R}^2) \geq 4$
Coloring the Plane: an Upper Bound
Coloring the Plane: an Upper Bound

Also, \( \chi(\mathbb{R}^2) \leq 7 \)
Fractional Coloring

Like coloring, but we can color a vertex part red and part blue.

\begin{align*}
2,4 & \quad 3,5 \\
1,4 & \quad 2,5 \\
1,3 & \quad 2,4,6 \\
3,5,7 & \quad 2,4,7 \\
2,5,7 & \quad 1,3,6 \\
2,4,7 & \quad 3,5,7 \\
\end{align*}

Weight \( w \in [0,1] \) for each ind. set \( I \) so each vert in sets that sum to 1; min sum of weights is \( \chi_f(G) \); weights in \( \{0,1\} \) give \( \chi(G) \).

Prop. \( \chi_f(G) \geq |V(G)| \alpha(G) \).

\begin{align*}
|V(G)| &= \sum_{v \in V(G)} \sum_{I \ni v} w_I \\
&= \sum_{I \in I} w_I |I| \\
&\leq \alpha(G) \sum_{I \in I} w_I = \alpha(G) \chi_f(G).
\end{align*}

When \( G \) is vertex transitive, \( \chi_f(G) = |V(G)| \alpha(G) \).
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![Graph diagram with vertices labeled 1, 2, 3, 4, 5, 6, and 7. Edges connect vertices as follows: 1-2, 1-4, 2-4, 2-5, 3-5, 3-7, 4-7, 5-7, and 6-7.]

- Weight $w_I \in [0, 1]$ for each independent set $I$ so each vertex in sets that sum to 1;
- Minimum sum of weights is $\chi_f(G)$;
- Weights in $\{0, 1\}$ gives $\chi(G)$.

Prop. $\chi_f(G) \geq |\text{V}(G)| \alpha(G)$.

$|\text{V}(G)| = \sum_{v \in \text{V}(G)} \sum_{I \ni v} w_I = \sum_{I \in I} w_I |I| \leq \alpha(G) \sum_{I \in I} w_I = \alpha(G) \chi_f(G)$.

When $G$ is vertex transitive, $\chi_f(G) = |\text{V}(G)| \alpha(G)$. 
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$\chi_f(C_5) \leq \frac{5}{2}$
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Prop. $\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)}$. 
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$|V(G)|$
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\[
|V(G)| = \sum_{v \in V} \sum_{I \ni v \forall v} w_I = \sum_{I \in \mathcal{I}} w_I |I|
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\[
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\]

When \( G \) is vertex transitive, \( \chi_f(G) = \frac{|V(G)|}{\alpha(G)} \).
Recall $\chi_f(G) \geq |V(G)|/\alpha(G)$.

More generally:

- $\mu: V(G) \to \mathbb{R} \geq 0$ is a weight function
- $|V(\mu(G))| := \sum_{v \in V} \mu(v)$ and $\alpha(\mu(G)) := \max_{I \in I} \sum_{v \in I} \mu(v)$

For every $\mu$, $\chi_f(G) \geq |V(\mu(G))|/\alpha(\mu(G))$. 
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$|V_\mu(G)| := \sum_{v \in V} \mu(v)$ and $\alpha_\mu(G) := \max_{I \in I} \sum_{v \in I} \mu(v)$

For every $\mu$, $\chi_f(G) \geq |V_\mu(G)|/\alpha_\mu(G)$. 
Recall \( \chi_f(G) \geq |V(G)|/\alpha(G) \).
Fractional Coloring, II

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A Computational Approach

Goal:
Find unit distance $H$ with $\chi_f(H) > 3.5$.

Idea:
Recall $\chi_f($spindle$) = 3.5$.
Find graph with many spindles that interact; at least one colored suboptimally. Core vertices from triangular lattice; attach many spindles; solve for best weights.

Core weights above, spindle weights 1, total weight: 51 + 45 = 96. Max independent set weight: 27. $\chi_f(H) \geq \frac{96}{27} = 3.5555...$
A Computational Approach

**Goal:** Find unit distance $H$ with $\chi_f(H) > 3.5$. 
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![Graph Diagram]

Core weights above, spindle weights 1, total weight: 51 + 45 = 96. Max independent set weight: 27. $\chi_f(H) \geq 96/27 = 3.5555$...
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![Diagram of a graph with core vertices and spindles]
A Computational Approach

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**Idea:** Recall $\chi_f$ (spindle) = 3.5. Find graph with many spindles that interact; at least one colored suboptimally. Core vertices from triangular lattice; attach many spindles;
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Core weights above, spindle weights 1, total weight: $51 + 45 = 96$. Max independent set weight: 27.

$$\chi_f(H) \geq \frac{96}{27} = \frac{32}{9} = 3.5555\ldots$$
Bigger Cores

\[
\chi_f \geq 168.47 
\approx 3.5744 
\]

\[
\chi_f \geq 491.137 
\approx 3.5839 
\]
Bigger Cores

Spindle weight 1 gives

\[ \chi f \geq \frac{168}{47} \approx 3.5744 \]
Bigger Cores

Spindle weight 1 gives
\[ \chi_f \geq \frac{168}{47} \approx 3.5744 \]

Spindle weight 2 gives
\[ \chi_f \geq \frac{491}{137} \approx 3.5839 \]
Our Biggest Core

\[ \chi_f \geq 1732.481 \approx 3.6008 \]
Spindle weight 3 gives \( \chi_f \geq \frac{1732}{481} \approx 3.6008 \)
A “By Hand” Approach

Big Idea:
- Extend same approach to entire plane.
  - Core is entire triangular lattice.
  - Use all possible spindles in 3 directions.
  - Each core vertex: weight 12
  - Each spindle vertex: weight 1
  - Avoid $\infty$: consider limit of bigger and bigger cores.

Core vertices: $M$
Total vertices: $M + 9$ $M - o(M)$
Total weight: $12M + 9$ $M - o(M) = 21M - o(M)$

Lem: Each independent set hits weight at most $6M$.

Pf: Next slide.

$\chi_f \geq \frac{21M}{6M} = \frac{7}{2} = 3.5$.
A “By Hand” Approach

**Big Idea:** Extend same approach to entire plane.
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- Total vertices: $M + 9$
- Total weight: $12M + 9 - o(M)$
  - Total weight: $21M - o(M)$

**Lemma:** Each independent set hits weight at most $6M$.

**Proof:** Next slide.

$$\chi_f \geq \frac{21M}{(6M)} = \frac{7}{2} = 3.5$$
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Total weight: \( 12M + 9M - o(M) = 21M - o(M) \)
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Core vertices: $M$
Total vertices: $M + 9M - o(M)$
Total weight: $12M + 9M - o(M) = 21M - o(M)$

**Lem:** Each independent set hits weight at most $6M$.

**Pf:** Next slide.

$$\chi_f \geq \frac{21M}{(6M)} = \frac{7}{2} = 3.5$$
The Discharging

Given independent set $I$, discharge weight of $I$ as follows:

$(R1)$ Each core vertex in $I$ gives $1$ to each core nbr

$(R2)$ Each spindle vertex in $I$ splits its weight equally between the core vertices incident to its spindle that are not in $I$

Final weight on core vertices:

- $\sum_{v \in I} \mu(v) \leq 6M$, so $\chi_f \geq \frac{21M}{6M} = 3.5$. 

- $\sum_{v \in I} \mu(v) = 6M$.
The Discharging

Given independent set $I$, discharge weight of $I$ as follows:

(R1) Each core vertex in $I$ gives 1 to each core nbr

\[
\begin{align*}
\text{Final weight on core vertices:} & \\
\text{▶ in $I$:} & \quad 12 - 6(1) = 6 \\
\text{▶ 3 nbrs in $I$:} & \quad 0 + 3 + \frac{6}{2} = 6 \\
\text{▶ 2 nbrs in $I$:} & \quad 0 + 2 + \frac{4}{2} + 2 = 6 \\
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\end{align*}
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Now $\sum_{v \in I} \mu(v) \leq 6M$, so $\chi_f \geq \frac{21M}{6M} = 3.5$. 
The Discharging

Given independent set $I$, discharge weight of $I$ as follows:

(R1) Each core vertex in $I$ gives 1 to each core nbr

(R2) Each spindle vertex in $I$ splits its weight equally between the core vertices incident to its spindle that are not in $I$

Final weight on core vertices:
- $\triangledown I$: $12 - 6(1) = 6$
- $3$ nbrs in $I$: $0 + 3 + 6 = 6$
- $2$ nbrs in $I$: $0 + 2 + 4 = 6$
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$$\chi_f \geq \frac{21M}{6M} = 3.5$$
A Hint of a Better Bound

To improve bound:

- Optimize the ratio of core weight and spindle weight

Now compute the final weight, averaged over each tile.

\[ \chi_f(R_2) \geq \frac{105}{29} \approx 3.6207 \]
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\[ \chi_f(R) \geq 10^{5.29} \approx 3.6207 \]
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A Tiling for a Better Bound
Summary

$\chi(R^2) \leq 7$; bounds unchanged since 50s

Lower bounds for $\chi_f(R^2)$ come from unit distance graphs

Moser spindle shows $\chi_f(R^2) \geq 3.5$

Main tool: $\chi_f \geq |V(G)|/\alpha(G)$

Weighted: $\chi_f \geq |V_\mu(G)|/\alpha_\mu(G)$

Fisher–Ullman proved $\chi_f(R^2) \geq 3.555$

Core from triangular lattice

Attach many spindles (all with weight 1)

Max. weight sum so no independent set hits more than 27 (solve LP)

Now $\chi_f(R^2) \geq 96/27 = 32/9 = 3.555$

Bigger cores give $\chi_f \geq 3.6008$

By hand: consider entire triangular lattice (via limits)

Core with $M$ vertices: total weight $21M$

Max independent set hits weight $6M$ (via discharging)

This proves $\chi_f(R^2) \geq (21M)/(6M) = 3.5$

Average over larger subsets of vertices: $\chi_f(R^2) \geq 3.6206$
Summary

- $4 \leq \chi(\mathbb{R}^2) \leq 7$
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