Painting Squares with $\Delta^2 - 1$ shades

Daniel W. Cranston
Virginia Commonwealth University
dcranston@vcu.edu

Joint with Landon Rabern
Slides available on my webpage

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Coloring Squares

Thm [Brooks 1941]: If ∆(G) ≥ 3 and ω(G) ≤ ∆(G), then χ(G) ≤ ∆(G) ≤ ∆(G)^2.

If G is connected and not Petersen, then ω(G) ≤ 8.

Conj [C.–Kim '08]: If G is connected, not a Moore graph, and ∆(G) ≥ 3, then χ_ℓ(G) ≤ ∆(G)^2 − 1.

Thm [C.-Rabern '14+]: If G is connected, not a Moore graph, and ∆(G) ≥ 3, then χ_ℓ(G) ≤ ∆(G)^2 − 1.
Coloring Squares

**Thm [Brooks 1941]**: If $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$ then $\chi(G) \leq \Delta(G)$.
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**Thm** [C.–Kim '08]: If $\Delta(G) = 3$ and $\omega(G^2) \leq 8$, then $\chi(G^2) \leq 8$. 

The Finale

So for $k = 7$ our desired Moore graph exists and is unique!
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**Thm [C.–Kim ’08]:** If $\Delta(G) = 3$ and $\omega(G^2) \leq 8$, then $\chi_l(G^2) \leq 8$. 

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![Graphs and diagrams related to coloring squares.](image-url)
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Related Problems

Wegner's (Very General) Conjecture [1977]:
If $G_k$ is the class of all graphs with $\Delta \leq k$, then for all $k \geq 3$, $d \geq 1 = \max_{G \in G_k} \chi(G)$

$\max_{G \in G_k} \omega(G)$.

Our result implies Wegner's conj. for $d = 2$ and $k \in \{4, 5\}$.

Borodin–Kostochka Conjecture [1977]:
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Our result implies B–K conj. for $G_2$ when $G$ has girth $\geq 9$. 
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Key Idea: $d_1$-choosable graphs
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**Def:** A graph $G$ is $d_1$-choosable if it has an $L$-coloring whenever $|L(v)| = d(v) - 1$ for all $v \in V(G)$. 

**Lem:** Minimal c/e $G_2$ contains no induced $d_1$-choosable subgraph $H$.

**Pf:** Color $G_2 \setminus V(H)$ by minimality. Consider a vertex $v \in V(H)$. Its number of colors available is at least $\Delta^2 - 1 - (d_{G_2}(v) - d_H(v)) \geq \Delta^2 - 1 - (\Delta^2 - d_H(v)) = d_H(v) - 1$. Extend coloring to $V(H)$, since $H$ is $d_1$-choosable.

Where to find $d_1$-choosable subgraph?
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![Diagram showing $G^2$ with $H$ as a subset, illustrating the concept of $d_1$-choosable graphs.](image-url)
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Where to find $d_1$-choosable subgraph?
Proof Outline

Consider a shortest cycle $C$ in $G$.

▶ 3-cycle: $d_G(v) \leq \Delta^2 - 2$ for each $v$ on $C$.

▶ 4-cycle: $d_G(v) \leq \Delta^2 - 1$ for each $v$ on $C$.

▶ 6-cycle: $C_6$ is 4-regular and 3-choosable.

▶ 7-cycle: Let $H$ be $C +$ pendant edge. Now since $G$ has no shorter cycles, $G_2[V(H)] = H_2$ (no extra edges). Use Alon–Tarsi Theorem to prove $H_2$ is $d_1$-choosable.

▶ 8+ cycle: similar but may need two pendant edges.

▶ 5-cycle: structural analysis to find $d_1$-choosable subgraph.

How do we prove that ($cycle + pendant edge$)$_2$ is $d_1$-choosable?
Proof Outline

Consider a shortest cycle $C$ in $G$. 

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- ▶ 6-cycle: $C^{2}_{6}$ is 4-regular and 3-choosable.
- ▶ 7-cycle: Let $H$ be $C^{2} +$ pendant edge. Now since $G$ has no shorter cycles, $G^{2}[V(H)] \approx H^{2}$ (no extra edges). Use Alon–Tarsi Theorem to prove $H^{2}$ is $d_{1}$-choosable.
- ▶ 8+ cycle: similar but may need two pendant edges.
- ▶ 5-cycle: structural analysis to find $d_{1}$-choosable subgraph.

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- 7-cycle: Let $H$ be $C +$ pendant edge. Now since $G$ has no shorter cycles, $G_2[V(H)] \sim H^2_2$ (no extra edges). Use Alon–Tarsi Theorem to prove $H^2_2$ is $d_1$-choosable.

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Now since $G$ has no shorter cycles, $G^2[V(H)] \sim H^2$ (no extra edges).

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- **8+ cycle:** similar but may need two pendant edges.

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How do we prove that (cycle + pendant edge) $^2$ is $d_1$-choosable?
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Use Alon–Tarsi Theorem to prove $H^2$ is $d_1$-choosable.
Proof Outline

Consider a shortest cycle $C$ in $G$.

- **3-cycle:** $d_{G^2}(v) \leq \Delta^2 - 2$ for each $v$ on $C$.
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[Diagram of a graph with cycles and pendant edges]
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How do we prove that $(\text{cycle } + \text{ pendant edge})^2$ is $d_1$-choosable?
Alon–Tarsi to prove $d_1$-choosability

**Alon–Tarsi:** For a digraph $\vec{D}$, if $|EE(\vec{D})| \neq |EO(\vec{D})|$, then $\vec{D}$ is $f$-choosable, where $f(v) = 1 + d_\vec{D}(v)$ for all $v$. 
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![Graph diagram](image.png)
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![Graph Diagram]

**Lemma** If $\vec{D}_n$ is the square of $C_n$, with all edges oriented clockwise, then $|EE(\vec{D}_n)| - |EO(\vec{D}_n)|$ only depends on $n \pmod{3}$. 


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\[ \begin{array}{ccccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \ldots & n \\
\end{array} \quad \leftrightarrow \quad \begin{array}{ccccccc}
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![Parity-reversing bijections diagram]
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A Gallery of $d_1$-choosable graphs
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(a) $EE=30$, $EO=28$

(b) $EE=108$, $EO=107$

(c) $EE=88$, $EO=87$

(d) $EE=512$, $EO=515$

(e) $EE=751$, $EO=750$

(f) $EE=1097$, $EO=1096$
In Summary

Main Theorem: If $G$ is connected and not Petersen, Hoffman–Singleton, or a Moore graph with $\Delta = 57$, then $\chi_p(G^2) \leq \Delta^2 - 1$.

Why do we care?
▶ Solves conjecture of Cranston–Kim, even for paintability.
▶ Verifies Wegner's Conjecture for $d = 2$ and $k \in \{4, 5\}$.
▶ Verifies Borodin–Kostoch Conj. for $G^2$ when $girth(G) \geq 9$.

Key idea: $G^2$ can't contain induced $d_1$-paintable subgraph.
▶ Where is one?
Shortest cycle in $G +$ few pendant edges.

Main tool: Alon–Tarsi Theorem (for paintability)
▶ Neat trick: Don't count $|EE|$ and $|EO|$, just $|EE| - |EO|$.
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- Serves as a solution to the Cranston–Kim conjecture, even for paintability.
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