Graphs with $\chi = \Delta$ have big cliques

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Joint with Landon Rabern
Slides available on my webpage

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If $\Delta \geq 9$ and $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$. 

Why $\Delta \geq 9$?
$\Delta = 8$, $\omega = 6$, $\alpha = 2$ 
$\chi = \lceil \frac{15}{2} \rceil = 8$

Why $\Delta - 1$?
$K_{t-4}$ where $\Delta = t$, $\omega = t - 2$ 
$\chi = (t - 4) + 3 = t - 1$
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Introduction

What do we know?

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  - then $\omega \geq \Delta - 4$ for all $\Delta$
Main Theorem

**Def:** A hitting set is independent set intersecting every maximum clique.
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Proof: Let $G$ be minimal counterexample. $\Delta \geq 14$ by Lemma 2.
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**Proof:** Let $G$ be minimal counterexample. $\Delta \geq 14$ by Lemma 2. If $\omega = \Delta - 4$, then let $I$ be a hitting set expanded to be a maximal independent set; otherwise let $I$ be any maximal independent set.
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If $\omega = \Delta - 4$, then let $I$ be a hitting set expanded to be a maximal independent set; otherwise let $I$ be any maximal independent set.

- If $\Delta(G - I) \leq \Delta(G) - 2$, then win by Brooks’ Theorem.
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- If $\Delta(G - I) = \Delta(G) - 1$, then $G - I$ is a smaller counterexample, contradiction!
Random Hitting Sets

**Lovász Local Lemma:** Suppose we do a random experiment. Let $\mathcal{E} = \{E_1, E_2, \ldots\}$ be a set of bad events such that
- $\Pr(E_i) \leq p < 1$ for all $i$, and
- each $E_i$ is mutually independent of all but $d$ events.

If $4dp \leq 1$, then with positive probability no bad events occur.
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\( E_{uv} \) is independent of all but \( 2k(\Delta - (k - 1)) = 20k \) events. Finally, \( 4(20k)k^{-2} \leq 1 \iff k \geq 80 \).
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**Def:** A Mozhan Partition of a graph $G$ with $\Delta = 13$ is a partition of $V$ into clubhouses $V_1, \ldots, V_4$ and a vertex $v$ with certain properties.
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**Lem:** Every $\Delta$-critical graph with $\Delta = 13$ has a Mozhan partition.

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For $w \in V(R)$ and $j \in \{1, \ldots, 4\}$:
- If $d_{V_j}(w) = 3$, then $G[V_j + w]$ has a $K_4$ component.
- If $w$ has 2 neighbors in club $S$ of clubhouse $V_i$, then $\chi(S + w) = 4$. 

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**Def:** A Mozhan Partition of a graph $G$ with $\Delta = 13$ is a partition of $V$ into clubhouses $V_1, \ldots, V_4$ and a vertex $v$ with certain properties. For each $V_i$, components of $G[V_i]$ are clubs meeting in clubhouse $V_i$.

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- All other clubs are 3-colorable.
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**Lem:** Every $\Delta$-critical graph with $\Delta = 13$ has a Mozhan partition.
The Vertex Shuffle

Lemma 2: If \( G \) has \( \chi = \Delta = 13 \), then \( G \) has a \( K_{10} \).
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**Lemma 2:** If $G$ has $\chi = \Delta = 13$, then $G$ has a $K_{10}$.

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What next?

The four-colour theorem is the tip of the iceberg, the thin end of the wedge, and the first cuckoo of Spring. –William Tutte

Reed's Conjecture:

\[ \chi \leq \left\lceil \omega + \Delta + 1 \right\rceil \]

Theorem (Reed):

There exists \( \epsilon > 0 \) such that

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Conjectured that \( \epsilon = \frac{1}{2} \) works.
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**Idea:** a partial coloring minimizing number of edges within clubhouses.

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