COLORING AND LABELING PROBLEMS ON GRAPHS

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Computer Science
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2007

Urbana, Illinois

Abstract

This thesis studies both several extremal problems about coloring of graphs and a labeling problem on graphs.

We consider colorings of graphs that are either embeddable in the plane or have low maximum degree. We consider three problems: coloring the vertices of a graph so that no adjacent vertices receive the same color, coloring the edges of a graph so that no adjacent edges receive the same color, and coloring the edges of a graph so that neither adjacent edges nor edges at distance one receive the same color. We use the model where colors on vertices must be chosen from assigned lists and consider the minimum size of lists needed to guarantee the existence of a proper coloring.

More precisely, a *list assignment function L* assigns to each vertex a list of colors. A *proper L-coloring* is a proper coloring such that each vertex receives a color from its list. A graph is *k-list-colorable* if it has an *L*-coloring for every list assignment *L* that assigns each vertex a list of size *k*. The *list chromatic number* $\chi_l(G)$ of a graph G is the minimum k such that G is k-list-colorable. We also call the list chromatic number the *choice number* of the graph. If a graph is *k-list-colorable*, we call it *k-choosable*.

The *elements* of a graph are its vertices and edges. A *proper total coloring* of a graph is a coloring of the elements so that no adjacent elements and no incident elements receive the same color. The *total list-chromatic number* is the minimum list size that guarantees the existence of a proper total coloring. We give a linear-time algorithm to find a proper total coloring from lists of size $2\Delta(G) - 1$. When $\Delta(G) = 4$, our algorithm improves the best known upper bound. When $\Delta(G) \in \{5, 6\}$ our algorithm matches the best known upper bound and runs faster than the best previously known algorithm.

The *square* of a graph G is the graph obtained from G by adding the edge xy whenever the distance between x and y in G is 2. We study the list chromatic numbers of squares of subcubic graphs; a graph is *subcubic* if it has maximum degree at most 3. We show that the square of every subcubic graph other than the Petersen graph is 8-list-colorable. For planar graphs with large girth, we use the discharging method to improve this upper bound. We show that the square of a planar subcubic graph with girth at least 7 is 7-list-colorable. We show that the square of a planar subcubic graph with girth at least 9 is 6-list-colorable. In each case we give linear-time algorithms to construct the colorings from the assigned lists.

The *strong edge-chromatic number* of a graph is the minimum number of colors needed to color the edges so that no two edges on a path of length at most 3 receive the same color. Erdős and Neśetril conjectured that when $\Delta(G) = 4$, the strong edge-chromatic number is at most 20; they gave a construction requiring 20 colors. The previous upper bound was 23, due to Horak. We improve this upper bound to 22.

We study the list edge-chromatic numbers of planar graphs. A graph is *k-edge-choosable*, if its line graph $L(G)$ is *k*-choosable. We call the choice number of the line graph $L(G)$ the *edge choice number* of

G. A *kite* is the union of two 3-cycles that share an edge. We show that if a planar graph has no kite (as a subgraph) and has maximum degree at least 9, then its list edge-chromatic number equals its maximum degree. We also show that if a planar graph has no kite (as a subgraph) and has maximum degree at least 6, then the list edge-chromatic number is at most one more than the maximum degree; the optimal bound is at most one less than this.

A graph is (r,s)-*choosable* if whenever each vertex is given a list of r colors, we can choose a sublist of s colors for each vertex so that adjacent vertices receive disjoint sublists. A graph is G (r,s)-*edge-choosable* if its line graph $L(G)$ is (r,s)-choosable. Mohar [38] conjectured that all 3-regular graphs are $(7,2)$-edge-choosable. If true, this result would be tight. We show that all 3-edge-colorable graphs are $(7,2)$-edge-choosable; in addition, we show that many snarks are $(7,2)$-edge-choosable. In each case, we give a linear-time algorithm to construct the coloring from given lists.

The *sum choice number* of a graph is the minimum total weight of a positive integer valuation of its vertices such that the graph is L-colorable for any list assignment L that the size of the list for each vertex is the integer value given to that vertex. We generalize this idea to the *k-sum choice number*, which is the minimum sum of list sizes such that we can choose k colors for each vertex (from its list) so that the sets of colors assigned to adjacent vertices are disjoint. We determine the 2-sum choice number of paths and cycles; additionally we determine all list-size assignment functions that achieve the 2-sum choice number for paths and cycles.

A *labeling* of a graph is a bijective function from the set $\{1, 2, \ldots, |E|\}$ onto the edges of the graph. The sum of the labels on edges incident to a vertex v is the *vertex-sum* at *v*. A labeling is *antimagic* if the vertex-sums are distinct. Ringel [20] conjectured that every connected graph other than K_2 has an antimagic labeling. We prove that every regular bipartite graph other than a matching has an antimagic labeling.

Acknowledgements

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Chapter

Chapter 1

Introduction

Graph coloring is a model for partitioning problems. We seek to partition a set of objects into subsets that avoid violating constraints. We define a graph whose vertices are the objects; two vertices are joined by an edge if they are not allowed to be in the same set in the partition. We name the sets of the partition by colors; usually the colors are positive integers. An alternative phrasing is that a coloring of a graph is a function that assigns labels (colors) to the vertices.

Most often, we want to minimize the number of labels in a coloring that satisfies the constraints, where constraints forbid vertices from having the same label. Variations of the problem introduce more general constraints, restrictions on the colorings that may be considered, or other measures of the coloring to optimize.

1.1 Coloring Squares of Graphs

The *square* G^2 of a graph G is formed from G by adding the edge xy whenever the distance between vertices x and y in G is 2. The line graph $L(G)$ of a graph G has as its vertices the edges of G; two vertices of $L(G)$ are adjacent if their corresponding edges share an endpoint. In Chapters 3 and 4, we study problems of coloring the edges of a graph. This is equivalent to coloring the vertices of its line graph. Discussion of such problems is usually simpler in the language of coloring edges of the original graph. However, to understand the relationship of the different problems we consider in this thesis, it is useful to view these problems as coloring the vertices of the line graph.

We begin Chapter 2 with the problem of coloring the square of a line graph. The value of $\chi(L(G)^2)$ is bounded in terms of the maximum degree $\Delta(G)$ of G. Let $\sigma_k = \max_{G: \Delta(G)=k} \chi(L(G)^2)$. Erdös and Nesetril gave a construction that requires $\frac{5k^2}{4}$ colors when k is even and $\frac{5k^2-2k+1}{4}$ colors when k is odd; they conjectured that this is σ_k. It is easy to verify this conjecture for $k \leq 2$. Andersen [2] proved the conjecture for $k = 3$; $\sigma_3 = 10$. For $k = 4$, the conjectured value is 20. Horak [27] gave the previous best upper bound: $\sigma_4 \leq 23$. We prove that $\sigma_4 \leq 22$.

The total graph $T(G)$ of a graph G has as its vertices the "elements" (vertices and edges) of G; two vertices of $T(G)$ are joined by an edge if the corresponding elements are incident or adjacent. The incidence graph is bipartite, with the vertices of G forming one part and the edges of G forming the other part; two

vertices of the incidence graph are adjacent if the corresponding elements are incident in G. The total graph is the square of the incidence graph.

A *list assignment* is a function L that assigns to each vertex a list of colors (usually positive integers). Given a list assignment L, a graph G is L-colorable if G has a proper coloring such that each vertex receives a color from its assigned list. A graph is k-choosable if it is L-colorable for every function L that assigns to each vertex a list of size k. The *list chromatic number* or *choosability* of a graph G, denoted $\chi_l(G)$, is the minimum k such that G is k-choosable. Analogously, we define *list edge-assignment*, *L-edge-colorable*, *k-edge-choosable*, *list edge-chromatic number*, and *edge-choosability*.

In Section 2.2, we consider the problem of list-coloring total graphs. In particular, we seek an algorithm that works well for small maximum degree. For $\Delta(G) = 3$, Juvan, Mohar, and Skrekovski [32] proved that $\chi_l(T(G)) \leq 5$. For $\Delta(G) > 3$, the previous best upper bound was $\lfloor \frac{3}{2}\Delta(G) \rfloor + 2$, due to Borodin, Kostochka, and Woodall [5]. We give an algorithm that produces a proper coloring from lists of size $2\Delta(G) - 1$. When $\Delta(G) = 4$, this improves the bound of Borodin, Kostochka, and Woodall. When $\Delta(G) \in \{5, 6\}$ our bound matches theirs; however, our algorithm is simpler and runs in linear time, unlike theirs.

A graph is *subcubic* if its maximum degree is at most three. The third problem we study in Chapter 2 is list-coloring the square of a subcubic graph. Recently, Thomassen [41] proved that the chromatic number of the square of a planar subcubic graph is at most 7. In 2001, Kostochka and Woodall [35] conjectured that $\chi_l(G^2) = \chi(G^2)$ for every graph G. We begin by considering the square of every subcubic graph (not necessarily planar). The square of the Petersen graph is K_{10}, which requires 10 colors. However, we show that $\chi_l(G^2) \leq 8$ for every subcubic graph G that is not the Petersen graph.

Our technique is to choose colors for almost all of the vertices greedily. The maximum degree in the square of a subcubic graph can be as large as 9. We give an ordering in which each vertex (except for a few at the end of the ordering) preceeds at least two of its neighbors in G^2. When each vertex having at most 7 earlier neighbors, the greedy coloring uses at most 8 colors (adding 1 for the vertex itself). The main difficulty in proving the theorem is showing that we can color the last few vertices in the ordering (those that don't preceed at least two of their neighbors in G^2).

We also consider list-coloring the square of a subcubic planar graph with large girth. However, because those results use a different method, we defer them to Chapter 3.

1.2 Discharging

In Chapter 3, we prove results for two different coloring problems; all of our results use the "discharging method". The discharging method is a technique for proving structural properties of a graph in the presence of a global complexity bound such as a bound on the average vertex degree. Many such results guarantee that every graph in such a class contains at least one of a specified set of subgraphs with small vertex degrees.

For example, a well-known lemma of Wernicke states: If the minimum degree in a planar graph is 5, then the graph contains an edge uv such that $d(u) + d(v) \leq 11$. Borodin strengthened this result to prove that: If the minimum degree in a planar graph is 5, then the graph contains a triangle uvw such that $d(u) + d(v) + d(w) \leq 17$. The presence of these subgraphs with small degree-sum can then be used in inductive proofs of coloring results. The discharging method has been particularly successful when applied to planar graphs, where Euler's Formula yields a natural bound on the average vertex degree.

In 1964, Vizing [44, 45, 19] proved that every graph G satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. The most famous conjecture in list-coloring is the List Coloring Conjecture [4], which asserts that every graph G satisfies $\chi'_l(G) = \chi'(G)$. Häggkvist and Janssen [22] proved that $\chi'_l(G) \leq \Delta(G) + c\Delta(G)^{2/3}\log\Delta(G)$, where c is a constant greater than 0. Kostochka [34] proved that if all cycles in G are long enough relative to $\Delta(G)$, then $\chi'_l(G) \leq \Delta(G) + 1$.

There has been even more substantial progress on proving the List Coloring Conjecture for planar graphs. In 1990, Borodin [6] proved it for planar graphs with maximum degree at least 14. In 1997, Borodin, Kostochka, and Woodall [5] proved it for planar graphs with maximum degree at least 12. We consider planar graphs that have smaller maximum degree and avoid certain subgraphs. A *kite* is two 3-cycles sharing an edge. We show that the List Coloring Conjecture is true for planar graphs that have no kites and have maximum degree at least 9.

We also consider the weaker conjecture that $\chi'_l(G) \leq \Delta(G) + 1$ for every graph G; this is called Vizing's Conjecture. We prove Vizing's Conjecture for planar graphs that have no kites and have maximum degree at least 6. This improves results of Zhang and Wu and of Wang and Lih. Zhang and Wu [55] showed that Vizing's Conjecture is true for a planar graph G if $\Delta(G) \geq 6$ and G has no 4-cycle. Wang and Lih [47] showed that Vizing's Conjecture holds for a planar graph G if $\Delta(G) \geq 6$ and G has no two triangles sharing a vertex. In each case, the set of subgraphs that we forbid (a kite) is a strict subset of the set of subgraphs forbidden in the previous results. Hence, our results are stronger.

In Chapter 3, we further study the problem of list-coloring the square of a subcubic graph; here we consider planar subcubic graphs with large girth. We show that if G is subcubic, planar, and has girth at least 7, then $\chi_l(G^2) \leq 7$. We also show that if G is subcubic, planar, and has girth at least 9, then $\chi_l(G^2) \leq 6$.

1.3 (a,b)-choosability

In the paper in which Erdös, Rubin, and Taylor introduced choosability, they also introduced (a,b)-choosability. A graph is (a,b)-*choosable* if whenever each vertex is assigned a list of a colors, we can choose a subset of b colors for each vertex from its assigned list so that adjacent vertices receive disjoint subsets. Thus, k-choosability is exactly $(k,1)$-choosability.

In Chapter 4, we study $(7,2)$-edge-choosability of 3-regular graphs. A graph G is (a,b)-*edge-choosable* if its line graph $L(G)$ is (a,b)-choosable. For a fixed graph G and positive integer b, it is natural to ask what the minimum a is such that G is (a,b)-choosable.

In a Problem of the Month (a section of his website where he frequently posts open problems), Bojan Mohar [38] asked for the minimum r such that every 3-regular graph is $(r,2)$-edge-choosable. He conjectured that every 3-regular graph is (7,2)-edge-choosable. It is not difficult to show that every 3-regular graph is (8,2)-edge-choosable, using a generalization of Brooks' Theorem. Tuza and Voigt [43] proved that: If a connected graph G is not complete and not an odd cycle, then G is $(\Delta(G)m,m)$-choosable for all $m \geq 1$. Since the line graph of a 3-regular graph has maximum degree 4, this implies that every 3-regular graph is (8,2)-edge-choosable.

It is also not difficult to construct a 3-regular graph that is not (6,2)-edge-choosable. Form G by subdividing an edge of K_4. We see by inspection that G is not (6,2)-edge-colorable and thus is not (6,2)-

edge-choosable. Hence, any 3-regular graph that contains G is not (6,2)-edge-choosable. As a result, the conjecture that every 3-regular graph is (7,2)-edge-choosable is sharp if true.

We show that every 3-edge-colorable graph is (7,2)-edge-choosable. We also show that many 3-regular graphs that are not 3-edge-colorable are still $(7,2)$-edge-choosable. Our main tool is the following lemma.

Lemma 1.1. *Let $A = \{a_1, \ldots, a_k\}$ be a matching and $B = \{b_1, \ldots, b_k\}$ be an edge set such that b_i is incident to a_i and a_{i+1} but not to any other edge in A (the subscript indices are viewed modulo k). Let the list assigned to edge e be $L(e)$, with all the lists having the same size. It is possible to choose one color for each edge of A from its list so that a_i and a_{i+1} together use at most one color from $L(b_i)$.*

By careful repeated application of this lemma, we reduce the problem to choosing two colors for each edge on a collection of vertex-disjoint cycles; each edge of these cycles has either 3, 4, or 5 remaining colors available. Coloring a graph from vertex lists of unequal size has been studied by Isaak [29, 30], Heinold [25] and Berliner, Bostelmann, Brualdi, and Deaett [3]. In comparing various functions for the list sizes, they seek to minimize the sum of the list sizes for a function where *L*-colorings are guaranteed, calling the minimum value of the sum the *sum-choice number* of the graph. We are not aware of any past work on the more general version of the problem, where we want to choose more than one color for each vertex.

A *list size function f* for a graph assigns to each vertex a list size. A graph is (f,k)-*choosable* if whenever each vertex v is assigned a list of size $f(v)$, we can choose k colors for each vertex from its list so that adjacent vertices receive disjoint sets of colors. We will apply results on (f,k)-edge-choosability to the problem of $(7,2)$-edge-choosability of 3-regular graphs. In particular, we study the $(f,2)$-edge-choosabilty of paths and cycles. Because the line graph of a cycle (path) is also a cycle (path), we simply study the $(f,2)$-choosability of cycles and paths.

We show that for every cycle C_n there exists a list size function f with sum $4n$ such that C_n is $(f,2)$-choosable. Because of the application to the problem of $(7,2)$-edge-choosability, we are particularly interested in list size functions f with all sizes in $\{3,4,5\}$. We determine all such list size functions with sum $4n$ such that C_n is $(f,2)$-edge-choosable.

1.4   Antimagic Labelings

Antimagic labelings were introduced by Ringel in 1990. A *labeling* of a graph G is a bijection from $E(G)$ to $\{1, \ldots, |E(G)|\}$. For a fixed labeling, the *vertex-sum* at v is the sum of the labels used on edges incident to vertex v. A labeling is *antimagic* if the vertex-sums are distinct. We call a graph *antimagic* if it has an antimagic labeling.

In 1990, Ringel [20] conjectured that every connected graph other than K_2 is antimagic. The most significant progress on this problem is a result of Alon et al. [1], which states the existence of a constant c such that if the minimum degree in an n-vertex graph is at least $c \log n$, then the graph is antimagic. In this paper, we show that every regular bipartite graph (with degree at least 2) is antimagic. Our technique relies heavily on the Marriage Theorem.

A *1-factor* of a graph is a 1-regular spanning subgraph. The Marriage Theorem [52] says that every regular bipartite graph has a 1-factor. By induction, we can partition the edges of a regular bipartite graph

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into disjoint 1-factors. Throughout Chapter 5, we refer to the two parts of the bipartite graph as *A* and *B*, with each part of size *n*.

When two vertices have the same vertex-sum under a given labeling, we say that the vertices *conflict*. We view the process of constructing an antimagic labeling as resolving the "potential conflict" for each pair of vertices. When we have labeled a subset of the edges, we call the sum at each vertex a *partial sum*. Our general approach is to label all but a single 1-factor so that the partial sums in *A* are $0(\bmod\ 3)$, while the partial sums in *B* are not congruent to $0(\bmod\ 3)$. We label the final 1-factor with labels that are 0 modulo 3 so that we resolve all potential conflicts within *A* and within *B*.

1.5 Basic definitions for graphs

A *graph G* consists of a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*, such that each edge is an unordered pair of vertices. We call the pair of vertices that make up an edge the *endpoints* of that edge; if vertices *u* and *v* are the endpoints of an edge *e*, we say that *u* and *v* are *adjacent* and that they are each incident to *e*. Edges are *incident* if they have a common endpoint.

A *multigraph* is more general than a graph, allowing the edges to form a multiset of vertex pairs and allowing edges whose endpoints are not distinct. Edges having the same pair of endpoints are *multiple edges*. An edge whose endpoints are not distinct is a *loop*. A graph in the model defined above has neither loops nor multiple edges; in the context of a discussion of multigraphs, we may emphasize the absence of loops and multiple edges by calling a graph a *simple* graph.

The *degree* of a vertex *v*, denoted $d(v)$, is the number of edges that have the vertex as an endpoint, except that a loop counts twice toward the degree of its endpoint. If every vertex has degree *k*, the graph is *k-regular* (or simply *regular*). A 3-regular graph is *cubic*; a graph with maximum degree 3 is *subcubic*. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum and minimum degree of vertices in a graph *G*, respectively.

A *subgraph H* of a graph *G* is a graph *H* such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced* subgraph *H* of *G* is a maximal subgraph with the vertex set $V(H)$. The subgraphs formed by removing the the edge set E_1 and the vertex set V_1 are denoted by $G - E_1$ and $G - V_1$, respectively. If E_1 is a single edge or V_1 is a single vertex, we simply write $G - e$ or $G - v$, respectively.

Vertices *u* and *v* are *connected* if there exists a list of edges such that *u* is an endpoint of the first edge, *v* is an endpoint of the last edge, and each successive pair of edges share a common endpoint. The *distance* between *u* and *v* is the size of the smallest such list of edges. A graph is connected if every two vertices in it are connected. A *component* of a graph is a maximal connected subgraph.

When we draw a graph, we represent each vertex as a point and each edge as a line between its endpoints. If we can draw a graph *G* in the plane so that none of its edges intersect (except at their common endpoints), then we say that graph *G* is *planar*; we call such a drawing a *planar embedding*. We call a particular planar embedding of a planar graph a *plane graph*. The *faces* of a plane graph are the maximal connected regions of the plane not containing a vertex or a point along an edge of the embedding.

We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of a graph *G*.For a plane graph *G*, we use $F(G)$ to denote the set of faces and *f* to denote $|F(G)|$. The *length* of a face is the number of edges on the boundary of the face; if an edge appears twice along the boundary of a face, then the edge counts twice toward the length of the face. A face of length three is a *triangle*; if every face of a planar embedding is a

triangle, then the embedding is a *planar triangulation*. A *k*-vertex is a vertex of degree *k*; a *k*-face is a face of length *k*.

The *square G^2* of a graph *G* is formed from *G* by adding the edge *xy* whenever the distance between vertices *x* and *y* in *G* is 2. The *Petersen graph* has as its vertices the 2-element subsets of a 5-element set; two vertices are adjacent if the corresponding 2-element subsets are disjoint. Two non-adjacent vertices correspond to pairs whose union has three elements, and hence they have a unique common neighbor. Therefore, the square of the Petersen graph is K_{10}. In fact, the Petersen graph is the only cubic graph *G* such that G^2 is a complete graph on 10 vertices.

The *line graph H* of a graph *G* has as its vertices the edge set of *G*; two vertices are adjacent in *H* if their corresponding edges share an endpoint in *G*. The *elements* of a graph are its edges and vertices. The *total graph $T(G)$* of a graph *G* has as its vertices the elements of *G*; vertices of $T(G)$ are adjacent if their corresponding elements are incident in *G*.

A graph is a *path* if its vertices can be ordered so that vertices are adjacent exactly when they are successive. A graph is a *cycle* if its vertices can be placed on a circle so that vertices are adjacent exactly when they are successive on the circle. The *lengths* of paths and cycles are the sizes of their edge sets. If a graph contains some cycle, then its *girth* is the minimum length of its cycles.

A graph on *n* vertices is a *complete graph*, denoted K_n when it has *n* vertices, if the vertices are pairwise adjacent. A set of pairwise adjacent vertices in a graph is a *clique*; a set of pairwise nonadjacent vertices is an *independent set*.

A *coloring* of a graph *G* is an assignment of colors (often denoted by positive integers) to the vertices of *G*. A *proper coloring* is a coloring such that Similarly, an *edge-coloring* of *G* is a coloring of its line graph $L(G)$, and a *proper edge-coloring* of *G* is a proper coloring of $L(G)$; equivalently, we color $E(G)$ so that incident edges receive distinct colors. A *partial [edge]-coloring* of *G* is a proper [edge]-coloring of a subgraph of *G*.

A *k-coloring* is a coloring that uses *k* colors; if a graph has a proper *k*-coloring, then it is *k-colorable*. The *chromatic number* of a graph *G*, denoted $\chi(G)$, is the minimum *k* such that *G* is *k*-colorable. Analagously, we define edge-chromatic number, *k*-edge-coloring, and *k*-edge-colorable; we denote the edge-chromatic number of *G* by $\chi'(G)$.

Euler's Formula relates the numbers of faces, edges, and vertices in a connected plane graph: $f - e + n = 2$; this formula can easily be proved by induction on the sum of the numbers of vertices and edges. Euler's Formula is the basis of the discharging arguments that we study in Chapter 3. It also enables us to prove an upper bound on the average degree of a planar graph in terms of its girth. The maximum average degree of a graph *G*, denoted *mad(G)* is the maximum taken over all subgraphs *H* of the average degree of *H*. We often color a graph recursively; if our proof uses a bound on the average degree of *G*, then we need the same bound for all subgraphs of *G*.

Lemma 1.2. *If G is a planar graph with girth at least g, then $mad(G) < \frac{2g}{g-2}$.*

Proof: Every subgraph of *G* is a planar graph with girth at least *g*; hence, it suffices to prove this upper bound for the average degree of the full graph *G*. Since the sum of the degrees counts each edge twice, the average degree equals $2e/n$. Also, summing the lengths of the faces yields $2e \geq fg$, so $f \leq 2e/g$.

Substituting for f in Euler's Formula yields $e = n - 2 + f \leq n - 2 + 2e/g$, and solving for e yields $e \leq (n-2)g/(g-2)$. Hence the average degree is less than $2g/(g-2)$. $\qquad\square$

Chapter 2

Coloring Squares of Graphs

The best general bound for coloring a graph G (in terms of its maximum degree $\Delta(G)$) comes from Brooks' Theorem [52], which states that $\chi(G) \leq \Delta(G) + 1$ and that $\chi(G) \leq \Delta(G)$ unless G is a complete graph or an odd cycle. Erdős, Rubin, and Taylor [13] extended this result to list coloring by showing that $\chi_l(G) \leq \Delta(G) + 1$ and that $\chi_l(G) \leq \Delta(G)$ unless G is a complete graph or an odd cycle.

In this chapter, we give algorithms for three graph coloring problems. Each problem can be viewed as coloring (or list coloring) the square G^2 of a graph G. In each case, the graph G has a special structure that enables us to color (or list color) G^2 with fewer than $\Delta(G^2)$ colors. We will color almost all of the vertices greedily, then color the last few vertices more carefully. To ensure that we use fewer than $\Delta(G^2)$ colors, we will order the vertices so that at the time we greedily color a vertex v, at least two neighbors of v in G^2 will be uncolored.

We begin by reserving a connected nontrivial subgraph H to color more carefully after we color all the other vertices. For each vertex $v \notin V(H)$, we define the distance $d(v, H)$ to be the length of the shortest path in G from v to $V(H)$. We greedily color the vertices in decreasing order of $d(v, H)$.

Suppose that we are coloring an arbitrary vertex v that has distance at least 2 from H. Let w and x be the first two vertices after v on a shortest path from v to H. Both w and x will be uncolored, when we color v, since they have smaller distance to H. Since w and x are both adjacent to v in G^2, at most $\Delta(G^2) - 2$ neighbors of v (in G^2) are already colored when we color v. A similar argument holds for vertices having distance 1 from H, since H is nontrivial. That is, in G^2, v has at least two neighbors in $V(H)$. Therefore, we use at most $\Delta(G^2) - 1$ colors to greedily color $G^2 - V(H)$. The main difficulty in proving these theorems is choosing an appropriate subgraph H and showing that we can extend the coloring to $V(H)$ using at most $\Delta(G^2) - 1$ colors.

Although each of the problems we consider can be viewed as coloring the square G^2 of a graph G, we usually phrase the argument as coloring the graph G such that vertices at distance 2 receive different colors; this language allows us to highlight structural properties of the graph G that might otherwise be obscured.

2.1 Strong Edge-Coloring

A *proper edge-coloring* is an assignment of a color to each edge of a graph so that no two edges with a common endpoint receive the same color. A *strong edge-coloring* is a proper edge-coloring with the further property that no two edges with the same color lie on a path of length three. The *strong edge-chromatic number* is the minimum number of colors that allow a strong edge-coloring. In this section we consider the maximum possible strong edge-chromatic number as a function of the maximum degree of the graph. For other variations of the problem, we refer the reader to a brief survey by West [53] and a paper by Faudree, Schelp, Gyárfás, and Tuza [15].

We use $\Delta(G)$ to denote the maximum degree of a graph G. In the context of a particular graph G, we often write k to denote $\Delta(G)$. In 1985, Erdős and Nešetřil gave the following construction. Begin with a 5-cycle and expand each of two nonadjacent vertices into $\lfloor k/2 \rfloor$ nonadjacent vertices, each of which inherits all the neighbors of the original vertex; in the same way, expand each of the other three original vertices into $\lceil k/2 \rceil$ nonadjacent vertices. This graph has $\frac{5}{4}k^2$ edges when k is even and $\frac{1}{4}(5k^2 - 2k + 1)$ edges when k is odd; since it has no induced $2K_2$, all edges must receive distinct colors. Erdős and Nešetřil conjectured that for each k, the maximum strong-edge chromatic number of a graph with $\Delta(G) = k$ is exactly the number of edges in their construction. (The Erdős-Nešetřil construction for $k = 4$ is shown in figure (1a).) Chung et al. [10] later showed that for each k this is the unique largest graph with no induced $2K_2$.

Andersen [2] proved the conjecture for the case $k = 3$. In this section, we improve the result for the case $k = 4$. The best upper bound previously known was 23, proved by Horak [27]; we improve this upper bound to 22. Our proof is valid without change for multigraphs, but for simplicity we phrase it in the language of graphs. We use as colors the set $\{1, 2, \ldots, 22\}$ of integers from 1 to 22. A greedy coloring algorithm sequentially colors the edges, using the least color that is not already prohibited from use on an edge at the time the edge is colored. Figure (1b) shows that the color used on each edge e is restricted by colors on at most 24 other edges. We use the notation $R(e)$ to mean the edges that are colored before edge e that restrict the color on e.

For every edge e in any edge order, we have $|R(e)| \leq 24$. Thus, for every edge order, the greedy algorithm produces a strong edge-coloring that uses at most 25 colors. However, there is always some order of the edges for which the greedy algorithm uses exactly the minimum number of colors required. Our aim in this section is to construct an order of the edges such that the greedy algorithm uses at most 22 colors. Throughout this section, when we use the term coloring, we mean strong edge-coloring. Each component of G can be colored independently of other components, so we may assume that G is connected.

Let w be a fixed vertex of a graph G. Let $d(v, w)$ denote the distance from vertex v to w (i.e. the length of the shortest path with endpoints v and w). The distance from an edge $e = uv$ to w, denoted $d(e, w)$, is the minimum of the distances from u to w and from v to w. We say that an edge order is *compatible* with vertex w if e_1 precedes e_2 in the order only when $d(e_1, w) \geq d(e_2, w)$. Intuitively, we color all the edges at distance $i + 1$ (farther from w) before we color any edge at distance i (nearer to v). Similarly, if we specify a cycle C in the graph, we can define the $d(v, C)$ to be the length of the shortest path with v as one endpoint and the other endpoint in the set $V(C)$. We say an edge order is compatible with C if e_1 precedes e_2 in the order only when $d(e_1, C) \geq d(e_2, C)$. Finally, let $\chi'_S(G)$ be the minimum number of colors that allow a strong edge-coloring of G.
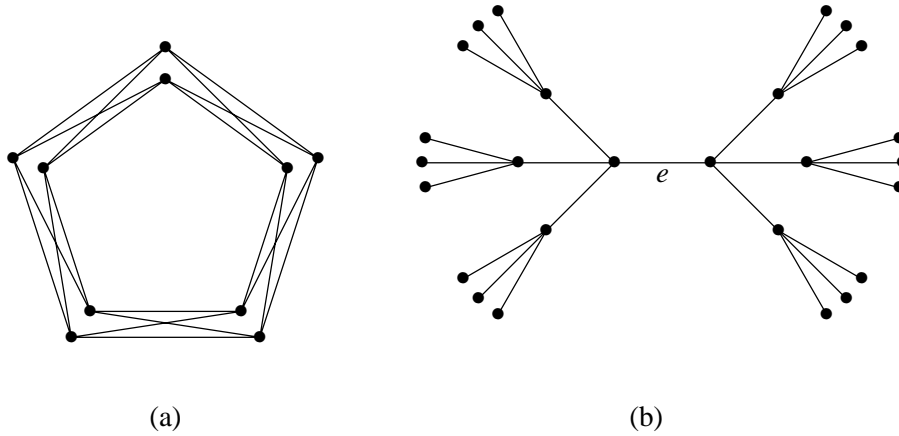
Figure 2.1. (a) The Erdős-Nešetřil construction for $\Delta(G) = 4$. This graph requires 20 colors. (b) The color used on each edge e is restricted by the colors on at most 24 other edges.

Lemma 2.1. *If $\Delta(G) = 4$, then G contains a vertex v such that $\chi'_S(G - v) \leq 21$. If $\Delta(G) = 4$, then G contains a cycle C such that $\chi'_S(G - E(C)) \leq 21$.*

Proof: We first consider coloring $G - v$, where $v \in G$. Greedily color the edges in an order that is compatible with v. Suppose we are coloring edge e that is not incident to v. Let u be the first vertex not in e along a shortest path from e to v. None of the four edges incident to u has been colored, since each edge incident to u has shorter distance from v. Thus, $|R(e)| \leq 24 - 4 = 20$.

To prove the case of coloring $G - E(C)$, we color the edges in an order compatible with C. The argument above holds for every edge not incident to C. If e is incident to C and $|V(C)| \geq 4$, then at least four edges whose colors restrict the color on e are edges of C; so again $|R(e)| \leq 24 - 4 = 20$. If e is incident to C and $|V(C)| = 3$, then by counting we see that the color on e is restricted by the colors on at most 23 other edges. The three uncolored edges of C imply that $|R(e)| \leq 23 - 3 = 20$. $\square$

Lemma 2.1 shows that if a graph has maximum degree 4, then we can color nearly all edges using at most 21 colors. In the rest of this section, we show that we can always finish the edge-coloring using at most one additional color. Our main result is that if $\Delta(G) = 4$, then $\chi'_S(G) \leq 22$. We begin by handling the easy cases: when the graph is not 4-regular, when the girth is at most 3, and when the girth is at least 6. We defer the other cases (when the graph is 4-regular and has girth 4 or 5) to the later part of this section.

Lemma 2.2. *Let G be a multigraph with maximum degree 4. If $\delta(G) < 4$ or G has girth less than 4, then $\chi'_S(G) \leq 21$.*

Proof: For any edge e incident to a vertex v of degree at most 3, there are at most 20 edges that can restrict the colors available on e. Therefore, a greedy coloring in an order compatible with v uses at most 21 colors.

If G contains a 3-cycle, let C be a 3-cycle. By Lemma 2.1, we can greedily color all edges except the edges of C using at most 21 colors. Each edge e of C has at most 20 edges that restrict the color used on e; hence we can finish the coloring with at most 21 colors. $\square$

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Lemma 2.3. *If G is 4-regular and has girth at least six, then $\chi'_S(G) \leq 22$.*

Proof: By Lemma 2.1, we can choose an arbitrary vertex v and greedily color all edges not incident to v using at most 21 colors. Now we recolor edges e_1, e_2, e_3, and e_4 (as shown in Figure 2) using color 22. Edges e_1, e_2, e_3, and e_4 can receive the same color since the girth of G is at least 6. Since each edge e incident to v is within distance 1 of each edges in $\{e_1, e_2, e_3, e_4\}$, at most 20 edges relevant to e have colors in $\{1, 2, \ldots, 21\}$, so we can finish the coloring greedily on the remaining four edges. $\square$
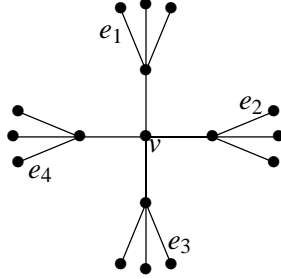
Figure 2.2. Vertex v has degree 4 and the girth of the graph is at least 6.

Lemma 2.3 proves Theorem 2.8 for 4-regular graphs with girth at least 6. In Sections 2.1.1 and 2.1.2, we consider 4-regular graphs with girths 4 and 5, respectively. We find pairs of edges that can receive the same color. In this case, even though $|R(e)| > 21$, because some edges in $R(e)$ do not receive distinct colors, we ensure that at most 22 colors are used.

2.1.1   4-regular graphs with girth four

Lemma 2.1 shows that we can color nearly all edges of the graph using 21 colors. Here we consider 4-regular graphs of girth four. We give an edge order such that the greedy coloring uses at most 22 colors; in some cases we precolor four edges prior to the greedy coloring. We use $A(e)$ to denote the set of colors available on edge e.

Lemma 2.4. *If G is 4-regular and has girth 4, then $\chi'_S(G) \leq 22$.*

Proof: Let C be a 4-cycle, with the 4 edges labeled c_i ($1 \leq i \leq 4$) in cyclic order. Label the pair of edges not on the cycle and adjacent to c_i and c_{i-1} is as a_i and b_i (all subscripts are mod 4). Let $S = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$. By Lemma 2.1, we greedily color all edges except the those on or incident to C. This uses at most 21 colors. If two edges of S share an endpoint not on C, they form a *bad pair*. The girth condition implies that that every bad pair must consist of one edge from $\{a_1, b_1\}$ and one edge from $\{a_3, b_3\}$ (or similarly one edge from $\{a_2, b_2\}$ and one edge from $\{a_4, b_4\}$).

Case 1: If the twelve uncolored edges contain at least two bad pairs, then we greedily color the edges in S. Each c_i has its color restricted by colors on at most 21 other edges, so $|A(c_i)| \geq 4$ for all i; thus we can finish by greedily coloring the four edges of C.
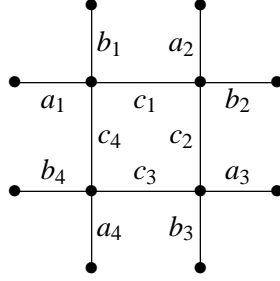
Figure 2.3. A 4-cycle in a 4-regular graph.

Case 2: Suppose the uncolored edges contain exactly one bad pair. For example, suppose edges a_2 and a_4 share an endpoint. Call edges a_1, b_1, a_3, and b_3 a *pack*.

Subcase 2.1: Suppose we can assign color 22 to two edges of the pack. Now we greedily color all edges except the edges of C. This uses at most 21 colors (Lemma 2.1). The color on each c_i is restricted by the colors on at most 22 edges. Since color 22 is used twice among these edges, each c_i satisfies $|A(c_i)| \geq 4$. So we can greedily finish the coloring.

Subcase 2.2: Suppose no pair of edges in the pack can receive the same color. This implies the existence of edges between each pair of nonadjacent edges of the pack. Call these four additional edges *diagonal* edges. Observe (by counting) that the color used on a diagonal edge is restricted by colors on at most 21 edges. So we can color the diagonal edges last in the greedy coloring. Thus we greedily color all edges except the four edges of C and the four diagonal edges (this uses at most 21 colors). Now we color the four edges of C (the four uncolored diagonal edges ensure there are enough colors available to color the edges of C). Lastly, we color the four diagonal edges.

Case 3: Finally, suppose that the uncolored edges contain no bad pairs. In this case we will greedily color almost all edges of the graph (Lemma 2.1), but must do additional work beforehand to ensure that after greedily coloring most of the edges each c_i will satisfy $|A(c_i)| \geq 4$. As above, call edges a_1, b_1, a_3, and b_3 a pack. Similarly, call edges a_2, b_2, a_4, and b_4 a pack.

Case 3.1: Suppose we can assign color 21 to two edges of one pack and assign color 22 to two edges of the other pack. We greedily color all edges but the four edges of C. Lemma 2.1 showed that a similar greedy coloring used at most 21 colors; however in Lemma 2.1 none of the edges were precolored. We adapt that argument to show that even in the presence of these four precolored edges a greedy coloring uses at most 22 colors. Lemma 2.1 argued there were at least four uncolored edges among those edges that restrict the color of the edge being colored, so $|R(e)| \leq 20$. The same argument applies in this case except that possibly one of the edges that was uncolored in Lemma 2.1 is now colored. Hence $|R(e)| \leq 21$ (this follows from the fact that the four uncolored edges in Lemma 2.1 were incident to the same vertex and in the present situation at most one precolored edge is incident to each vertex). Hence, the greedy coloring uses at most 22 colors. The color used on each c_i is restricted by the colors on at most 23 edges. Since colors 21 and 22 are each repeated among these edges, we see that each c_i satisfies $|A(c_i)| \geq 4$. So we can greedily finish the coloring.

Case 3.2: Suppose we cannot assign color 21 to two edges of one pack and assign color 22 to two edges of the other pack. If no two edges in a pack can receive the same color, this implies the existence of edges

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between each pair of nonadjacent edges of the pack. The color used on each of these four diagonal edges is restricted by the colors on at most 21 edges. As we did above, we greedily color all edges except the four edges of C and the four diagonal edges. Now we color the four edges of C, and lastly, we color the four diagonal edges. $\square$

2.1.2   4-regular graphs with girth five

Here we consider 4-regular graphs with girth five. As in the case of girth four, we color nearly all the edges by Lemma 2.1. Intuitively, if there are enough different colors available to be used on the remaining uncolored edges, we should be able to complete this coloring by giving each uncolored edge its own color. However, if there are fewer different colors available than the number of uncolored edges, this approach is doomed to fail. Hall's Theorem [52] formalizes this intuition. In the language of Hall's Theorem, we have m uncolored edges, and the set A_i denotes the colors available to use on edge i.

Theorem 2.5 (Hall's Theorem). *A family of sets $A_1, A_2, \ldots, A_m$ has a system of distinct representatives if and only if the union of any j of these sets contains at least j elements for all j from 1 to m.*

We define a *partial coloring* to be a strong edge-coloring except that some edges may be uncolored. Suppose that we have a partial coloring, with only the edge set T left uncolored. Let $A(e)$ be the set of colors available to color edge e. Then Hall's Theorem guarantees that if we are unable to complete the coloring by giving each edge its own color, there exists a set $S \subseteq T$ with $|S| > |\cup_{e \in S} A(e)|$. Define the *discrepancy*, $disc(S) = |S| - |\cup_{e \in S} A(e)|$.

Our idea is to color the set of edges with maximum discrepancy, then argue that this coloring can be extended to the remaining uncolored edges.

Lemma 2.6. *Let T be the set of uncolored edges in a partially colored graph. Let S be a subset of T with maximum discrepancy. Then a valid coloring for S can be extended to a valid coloring for all of T.*

Proof: Assume the claim is false. Since the coloring of S cannot be extended to $T \setminus S$, some set of edges $S' \subseteq (T \setminus S)$ has positive discrepancy (after coloring S). We show that $disc(S \cup S') > disc(S)$. Let R be the set of colors available to use on at least one edge of $(S \cup S')$. Let R_1 be the set of colors available to use on at least one edge of S. Let R_2 be the set of colors available to use on at least one edge of S' after the edges of S have been colored. Let $k = disc(S)$. Then $|S| = k + |R_1|$ and $|S'| \geq 1 + |R_2|$. Since S and S' are disjoint, we get

$$|S \cup S'| = |S| + |S'| \geq k + 1 + |R_1| + |R_2| > k + |R|.$$

The latter inequality holds since a color which is in $R \setminus R_1$ must be in R_2 and therefore we have $|R| = |R_1 \cup R_2| \leq |R_1| + |R_2|$. Hence

$$\text{disc}(S \cup S') = |S \cup S'| - |R| > k = \text{disc}(S).$$

This contradicts the maximality of disc(S). Hence, any valid coloring of S can be extended to a valid coloring of T. $\square$

Lemma 2.7. *If G is 4-regular and has girth 5, then $\chi'_S(G) \leq 22$.*

Proof: Let C be a 5-cycle, with the 5 edges labeled c_i ($1 \leq i \leq 5$) in cyclic order and the pair of edges not on the cycle and adjacent to c_i and c_{i-1} is labeled a_i and b_i (all subscripts are mod 5). Let $B = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$. Edge a_1 is at least distance 2 from at least one of edges a_3 and b_3; for if a_1 has edge e_1 to a_3 and edge e_2 to b_3 then we have the 4-cycle e_1, e_2, b_3, a_3. Thus (by possibly renaming a_3 and b_3) we can assume there is no edge between edges a_1 and b_3.
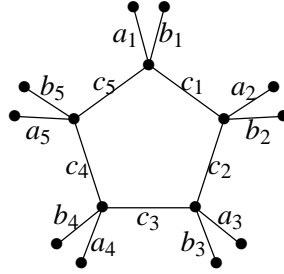
Figure 2.4. A 5-cycle in a 4-regular graph.

By repeating the same argument, we can assume there is no edge between the two edges of each of the following pairs: (a_1, b_3), (a_3, b_5), (a_5, b_2), and (a_2, b_4). Assign color 21 to edges b_1 and c_3 and assign color 22 to edges a_5 and b_2. Greedily color all edges except the edges of C and the edges in B. This uses at most 22 colors.

There are 11 uncolored edges; if we cannot assign a distinct color to each uncolored edge, then Hall's Theorem guarantees there exists a subset of the uncolored edges with positive discrepancy. Let S be a subset of the uncolored edges with maximum discrepancy. By counting the uncolored edges near each uncolored edge, we observe that if e is an edge of C, then $|A(e)| \geq 8$ and if e is an edge in B then $|A(e)| \geq 5$. We can assume that S contains some edge of C, since otherwise we can greedily color S (Lemma 2.1), then extend the coloring to the remaining uncolored edges (Lemma 2.6). Since $disc(S) > 0$ and $|A(e)| \geq 8$ for each edge of C, we have $|S|$ is 9, 10, or 11.

Case 1: Suppose $|S|$ is 9 or 10. Then since S is missing at most two uncolored edges, S contains at least one of the pair (a_1, b_3), the pair (a_2, b_4), and the pair (a_3, b_5). Since each edge in the pair satisfies $|A(e)| \geq 5$ and $|\cup_{e \in S} A(e)| \leq 9$, some color is available for use on both edges of the pair. Assign the same color to both edges. Note that each uncolored edge $e \in B$ satisfies $|R(e)| \leq 24 - 3 = 21$; so we can greedily color the remaining uncolored edges in B. Now if S contains the pair (a_1, b_3) or the pair (a_3, b_5) then color the edges of the 5-cycle in the order c_2, c_4, c_5, c_1; if S contains the pair (a_2, b_4) then color the edges of the 5-cycle in the order c_2, c_4, c_1, c_5.

Case 2: Suppose $|S|$ is 11 and that no color is available on both edges of any of the pairs (a_1, b_3), (a_2, b_4), and (a_3, b_5) (otherwise the above argument holds). Note that if $|A(c_1)| \geq 8$, $|A(a_4)| \geq 5$, and $|A(c_1) \cup A(a_4)| \leq |\cup_{e \in S} A(e)| \leq 10$, then $|A(c_1) \cap A(a_4)| \neq 0$. Assign the same color to c_1 and a_4; call it color x. Before color x was assigned to c_1 and a_4, it had been available on exactly one edge of each of the three pairs. Greedily color those three edges (none of the colors used on these three edges is color x). Now

the three remaining uncolored edges $e \in B$ each satisfy $|A(e)| \geq 3$, so we can greedily color them. Greedily color the three remaining edges in the order c_2, c_4, c_5. □

Theorem 2.8. *Any graph with maximum degree 4 has a strong edge-coloring with at most 22 colors.*

Proof: The theorem follows immediately from Lemmas 2.2, 2.3, 2.4, and 2.7. □

We note that it is straightforward to convert this proof to an algorithm that runs in linear time. We assume a data structure that stores all the relevant information about each vertex. Using breadth-first search, we can calculate the distance classes, as well as implement each lemma in linear time.

A natural question is whether it is possible to extend the ideas of this section to larger k. The best bound we could hope for from the techniques of this section is $2k^2 - 3k + 2$. It is straightforward to prove an analog of Lemma 2.1 that gives a strong edge-coloring of G that uses $2k^2 - 3k + 1$ colors except that it leaves uncolored those edges incident to a single vertex (however, the author was unable to prove an analog to the "uncolored cycle" portion of Lemma 2.1). If G a vertex of degree less than k, then by the analog of Lemma 2.1, G has a strong edge-coloring that uses at most $2k^2 - 3k + 1$ colors. Using the ideas of Lemma 2.3, we see that if G is k-regular and has girth at least 6, then G has a strong edge-coloring that uses $2k^2 - 3k + 2$ colors. Thus, to complete a proof for graphs with larger k, one must consider the case of regular graphs with girth 3, 4, or 5.

2.2 List-colorings of Total Graphs

In this section, we study the list chromatic number of "total graphs". When discussing these graphs, it is convenient to refer to the edges and vertices of a graph as its *elements*. The *total graph* $T(G)$ of a graph G then has as its vertices the elements of G, and two vertices of $T(G)$ form an edge in $T(G)$ if the corresponding elements of G are adjacent or incident in G.

An alternative construction of the total graph starts with the *subdivision graph* $S(G)$, which is formed by replacing each edge of G with a path of length 2 having the same endpoints as the original edge. Equivalently, the subdivision graph is the incidence graph of the incidence relation between vertices and edges in G. That is, it is a bipartite graph with partite sets $V(G)$ and $E(G)$, with $v \in V(G)$ adjacent to $e \in E(G)$ in $S(G)$ if v is an endpoint of e in G. Note that $V(T(G)) = V(S(G))$. In fact, the total graph is the square of the subdivision graph.

In most coloring problems, arguments for connected graphs or multigraphs apply to each component of a disconnected graph or multigraph, so when studying upper bounds on chromatic parameters for a family, it suffices to restrict our attention to connected members of the family.

Recall that the list chromatic number of a graph is its *choosability*, and that $\chi_l(H) \leq k$ is the meaning of *k-choosable*. If a total graph $T(G)$ is k-choosable, then we say that G is *totally-k-choosable*.

In most cases, our algorithm will greedily color all but a few edges and vertices of G; we generally call this uncolored subgraph H. This and the requirement of coloring the "elements" of a graph motivate the following definition.

For a graph G and a subgraph H, we abuse notation by writing $G \setminus H$ to denote the set of elements of G that are not elements of H; that is, the set $(V(G) - V(H)) \cup (E(G) - E(H))$.

For example, an edge uv may be present in $G \setminus H$ even if one or both of the vertices u and v are not present. When we produce a "total" coloring for $G \setminus H$, what we are actually doing is presenting a proper coloring for an induced subgraph of $T(G)$, the subgraph obtained by deleting the vertices of $T(G)$ that are elements of H. Thus it would be a bit easier to be completely precise with terminology by discussing the problem in the language of $\chi_l(T(G))$ or $\chi_l(S(G)^2)$, but we prefer to stick with the language of vertices and edges in G, because this is the source of the problem and because vertices and edges of G behave differently when we talk about bounds in terms of $\Delta(G)$.

Juvan, Mohar, and Škrekovski [32] showed that every graph with maximum degree 3 is totally-5-choosable. Skulrattanakulchai and Gabow [40] used their ideas to show that in this case a proper total coloring can be chosen from lists of size 5 in time that is linear in the number of vertices of G. We extend these ideas further, providing an algorithm when $\Delta(G) \geq 3$ that constructively chooses a proper total coloring from lists of size $2\Delta(G) - 1$ in linear time.

The best previous upper bound on the total choosability for $\Delta(G) > 3$ was $\lfloor \frac{3}{2}\Delta(G) + 2 \rfloor$, by Borodin et al. [6]. When $\Delta(G) = 4$, our result improves the upper bound. When $\Delta(G) \in \{3, 5, 6\}$, our algorithm matches the best known bound. However, our algorithm is significantly simpler and runs in linear time, unlike the algorithm of Borodin et al.

In Lemma 2.9, we greedily choose a total coloring for almost all elements of G, from lists of size $2\Delta(G) - 1$. The remainder of the section is devoted to extending the coloring to the remaining elements of G.

Lemma 2.9. *If C is a cycle in G, then $G \setminus E(C)$ is totally-$(2\Delta(G) - 1)$-choosable. If G contains a vertex v with $d(v) < \Delta(G)$, or G contains an edge with multiplicity at least 3, then G is totally-$(2\Delta(G) - 1)$-choosable.*

Proof: Let $k = \Delta(G)$. Colors must be chosen so that elements adjacent in $S(G)^2$ have distinct colors. Given a target set $R \subseteq V(S(G))$, for $x \in V(S(G))$ we define $f(x)$ to be the distance from x to R in $S(G)$, and we choose colors for all elements of $G \setminus R$ in decreasing order of f.

The idea is to reach x having previously colored fewer neighbors of x in $S(G)^2$ than the number of colors that are in the list available to x. Every element of $V(S(G))$ has at most $2k$ neighbors in $S(G)^2$ (a vertex of G can have k neighbors and k incident edges; an edge of G has two incident vertices and up to $2k - 2$ incident edges).

For this reason, $2k + 1$ is a trivial upper bound on $\chi_l(T(G))$. To improve the bound to $2k - 1$, it suffices to reserve two neighbors to be colored later. When we color x, the other vertices along a shortest path from x to R are not yet colored.

If $f(x) \geq 2$, then a shortest path to R has at least two elements after x, and hence a list of size $2k - 1$ suffices at x. When $f(x) = 1$, it suffices for x to have at least two neighbors in R.

Now consider $R = C$. We have $f(x) = 1$ precisely when x is a vertex of C, and x has two neighbors in $E(C)$, as desired.

For the second statement, let $R = \{v\}$, where v is a vertex of G with degree less than k or is a vertex incident to an edge with multiplicity at least 3. We must consider each edge x incident to v (that is, those with $f(x) = 1$) and also v itself ($f(v) = 0$).

If $d_G(v) < k$, then the neighborhood in $S(G)^2$ of an incident edge has size at most $2k - 1$, with v uncolored, so the list of size $2k - 1$ suffices. Furthermore, the neighborhood of v has size at most $2k - 2$, so we can choose a color for v at the end.

When v is incident to a multiple edge with at least three copies, we leave these three edges as the last to color before v. We continue to have two uncolored neighbors until the last copy of the multiple edge. Since the other copies are incident to it at both ends, it has at most $2k - 2$ neighbors. Similarly, the vertex v has at most $k - 2$ neighbors in G and hence at most $2k - 2$ in $S(G)^2$. $\qquad\square$

As mentioned earlier, we try to select colors greedily moving toward a remaining set elements where special arguments will complete the coloring chosen from the lists. Let H denote the remaining set of uncolored edges and vertices. We abuse terminology by refering to it sometimes as a subgraph of G and sometimes as a set of elements. In Lemmas 2.10 and 2.11, Juvan et al. [32] provided several choices for H and showed how to extend the coloring to H in each case.

For convenience, Juvan et al. define *halfedges* to be edges with only one endpoint. We use this term to describe an edge of H that has only one endpoint in H. Like an edge, a halfedge needs a color; the difference is that a halfedge in H has only one endpoint in H, so it has at most $\Delta(G) - 1$ incident edges in H.

Lemma 2.10. *([32]) Let H be a cycle with a halfedge attached to each vertex. If L is a list assignment for H such that*

$$|L(t)| \geq \begin{cases} 5, & \text{if } t \text{ is an edge,} \\ 4, & \text{if } t \text{ is a vertex,} \\ 2, & \text{if } t \text{ is a halfedge,} \end{cases}$$

then H admits an L-total-coloring.

We will show in Lemma 2.13 that a regular graph G with girth at least 5 contains an induced cycle whose vertices have a system of distinct neibhbors off the cycle. In this case, we will greedily color all the elements of G except for the cycle and the edges that match its vertices to these neighbors. With the edges of the matching treated as halfedges, we will apply Lemma 2.10 to finish the coloring (the details appear in Theorem 2.3.1).

The next two lemmas consider cases where G has shorter cycles. In each case we find a small subgraph H and greedily total-color $G \setminus H$; Lemmas 2.11 and 2.12 extend the colorings to H.

In Lemma 2.11 we refer to *thick* halfedges and *thin* halfedges. Both are halfedges as described above; the only difference is that thick halfedges will receive lists of size 3, whereas thin halfedges will receive lists of size 2. Thick halfedges always appear in pairs; they designate halfedges that are nonincident in H but correspond to incident edges in G.

Lemma 2.11. *([32]) Let H be isomorphic to one of the multigraphs in Figure 2.5. If L is a list assignment for H such that*

$$|L(t)| \geq \begin{cases} 5, & \text{if } t \text{ is an edge,} \\ 4, & \text{if } t \text{ is a vertex,} \\ 3, & \text{if } t \text{ is a thick halfedge} \\ 2, & \text{if } t \text{ is a thin halfedge,} \end{cases}$$

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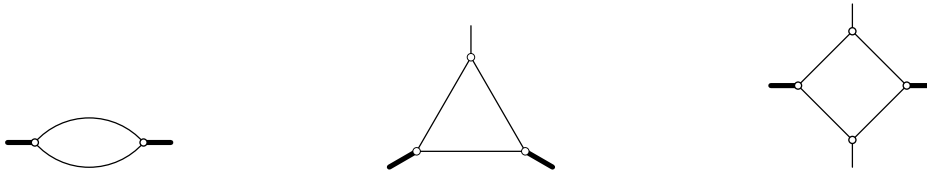
Figure 2.5. (a) A double edge with each endpoint incident to a thick halfedge. (b) A 3-cycle with two vertices incident to thick halfedges and a the third vertex incident to a thin halfedge. (c) A 4-cycle with two nonadjacent vertices incident to thick halfedges and the other two vertices incident to thin halfedges.

then H admits an L-total-coloring such that the two thick halfedges receive distinct colors.

In addition to these choices for H that Juvan et al. used in their proof that graphs with maximum degree 3 are totally-5-choosable, we need several additional choices to prove our generalization of their result. These appear in the following lemma. It should be noted that the result for the double-edge here would seem to imply the result for the double-edge in Lemma 2.11, where the lists on the halfedges are larger, but there the colors on the halfedges are required to be distinct.

Figure 2.6. (a) A double edge with each endpoint incident to a thin halfedge. (b) A complete graph on 4 vertices. (c) A complete bipartite graph with each vertex set of size 3.

Lemma 2.12. *Let H be K_4, $K_{3,3}$, or a double-edge with two incident halfedges (see Figure 2.6). If L is a list assignment for H such that*
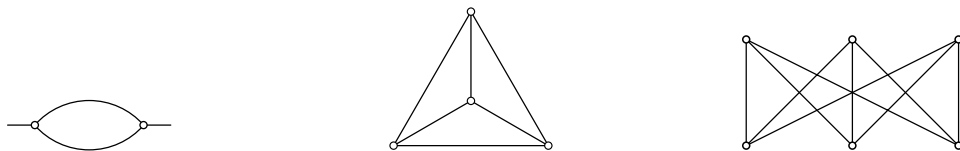
$$|L(t)| \geq \begin{cases} 5, & \text{if } t \text{ is an edge,} \\ 4, & \text{if } t \text{ is a vertex,} \\ 2, & \text{if } t \text{ is a halfedge,} \end{cases} \tag{2.1}$$

then H admits an L-total-coloring.

Proof: We may assume that the given inequalities on the list sizes hold with equality; otherwise we discard colors.

Suppose first that H has two vertices (v_1, v_2), two edges (e_1, e_2) having them as endpoints, and half-edges e_3 at v_1 and e_4 at v_2. Since $|L(v_1)| + |L(e_4)| > |L(e_1)|$, colors can be chosen for v_1 and e_4 from their lists so that at most one color is used from $L(e_1)$. Hence we can color v_1 and e_4 leaving lists of sizes $1, 2, 3, 4$ at e_3, v_2, e_2, e_1, respectively, and then extend the coloring in order to these elements.

For $H \cong K_4$, first greedily color the vertices in some order. Each edge e now has at least three colors in its remaining list $L'(e)$, since each edge lost at most one color to each endpoint. If we cannot select distinct colors from these lists, then by Hall's Theorem [52] on systems of distinct representatives there is a set S

of edges in H such that $|\bigcup_{e \in S} L'(e)| < |S|$. Since $|L'(e)| \geq 3$ for all e, we have $|S| \geq 4$. Among any four edges of K_4 there are two nonincident edges; call them e_1 and e_2 in S. Since $|L'(e_1) \cup L'(e_2)| < |S| \leq 6$, edges e_1 and e_2 have a common available color, c. Use color c on e_1 and e_2. This leaves at least two colors available on each remaining uncolored edge, and these edges form a 4-cycle. Every cycle of even length is 2-edge-choosable (Erdös, Rubin, and Taylor [13]). Therefore, the coloring extends to all of H.

For $H \cong K_{3,3}$, first greedily color the vertices in some order. Each edge e now has at least three colors in its remaining list $L'(e)$. Every bipartite multigraph H is $\Delta(H)$-edge-choosable (Galvin [18]). Therefore, again the coloring extends to all of H. $\qquad \square$

Our final lemma is structural. We will use it to obtain in any multigraph G a subgraph H such that we can extend a proper total coloring of $G \setminus H$ chosen from lists of size $2\Delta(G) - 1$ to a proper total coloring of G chosen from lists of that size.

We abuse terminology somewhat, accepting a double-edge as an "induced cycle" of length 2. If all adjacent pairs occur as edges with multiplicity 3, then in fact there is no induced cycle.

Lemma 2.13. *If G is a k-regular multigraph, then we can find in linear time an induced cycle that has length at most 4 or has no two vertices with a common neighbor off the cycle.*

Proof: If G has a multiple edge, then we can find one in linear time (since k is fixed). We accept two copies of an edge as an induced cycle of length 2. Hence we may assume that G is simple.

Choose any vertex v. Using breadth-first-search, find a shortest cycle D through v. By the choice of the cycle, vertices on D with a common neighbor outside D must be adjacent or have another common neighbor on D. With the common neighbor(s), they thus form a cycle in G of length at most 4. If there is no pair of vertices with such common neighbors, then D is the desired cycle. $\qquad \square$

By combining Lemmas 2.9 through 2.13, we prove our main result.

Theorem 2.14. *If G is a multigraph with maximum degree $\Delta(G)$, where $\Delta(G) \geq 3$, then G is totally-$(2\Delta(G) - 1)$-choosable. Furthermore, given lists of size $2\Delta(G) - 1$, we can choose a proper total coloring from the lists in linear time.*

Proof: If G is not $\Delta(G)$-regular or contains an edge with multiplicity at least 3, then Lemma 2.9 completes the proof. Hence we may assume that G is regular and has edge multiplicity at most 2.

Case 1: *G has an edge uv with multiplicity 2.)* Since $\Delta(G) \geq 3$, there are additional edges e_1 and e_2 incident to u and v, respectively. We view e_1 and e_2 as halfedges (thick if they have a common endpoint, thin otherwise). The resulting subgraph H is the first case in Figure 2.5 or Figure 2.6. Let C be the 2-cycle in H.

By Lemma 2.9, we can greedily color $G \setminus E(C)$; hence we can also stop the process before coloring e_1 and e_2 and any of $V(C)$. In order to apply Lemma 2.11 or Lemma 2.12 to complete the coloring, we must check that each uncolored vertex, edge, and halfedge in H has enough available colors remaining in its list.

Let $k = \Delta(G)$. A uncolored vertex (u or v) is incident to at most $k - 3$ colored edges and at most $k - 2$ colored vertices. With an initial list of size $2k - 1$, it thus has at least 4 remaining available colors. An uncolored edge (one of the uv edges) is incident to no colored vertices and to at most $k - 3$ colored edges

at each endpoint. Hence it has at least 5 remaining available colors. A thin halfedge (e_1 or e_2) is incident to one colored vertex; also it is incident to at most $k - 1$ colored edges at that end and at most $k - 3$ colored edges at the other end. Thus a thin halfedge has at least 2 remaining available colors. A thick halfedge has one additional available color, since it is incident to the other thick halfedge, which is uncolored.

Case 2: *G has no multiple edges.* Find an induced cycle C as described in Lemma 2.13. Since Lemma 2.9 allows us to choose colors for a proper total coloring of $G \setminus E(C)$, as in Case 1 we can stop the process before coloring any of $V(C)$ and also leave one uncolored edge off C incident to each vertex of C.

We need lower bounds on the numbers of remaining available colors in the lists for the uncolored elements. We have seen that when G is regular and simple, each element has exactly $2k$ neighbors in $T(G)$. To have r colors remaining available from a list of size $2k - 1$, it suffices to have $r + 1$ uncolored neighbors in $T(G)$. An edge of C neighbors two edges of C, two vertices of C, and two half-edges. A vertex of C has uncolored neighbors of similar counts, but only one half-edge. A half-edge neighbors one vertex and two edges of C. Hence the lists retain at least 5, 4, or 2 elements, respectively. Furthermore, when two of the half-edges are incident (hence thick), they have an additional uncolored neighbor in $T(G)$ (each other) and hence retain at least 3 available colors.

If the halfedges are non-incident, or C has length at least 5, or there is at most one pair of incident halfedges (nonconsecutive when C has length 4), then we have guaranteed that the lists of remaining available colors are large enough for Lemma 2.10 or Lemma 2.11 to guarantee completion of the coloring.

If $|V(C)| = 3$, then the remaining case is that the three uncolored halfedges have a common endpoint u. After the initial phase, we erase the color on u. Now the uncolored graph H is K_4. Each vertex or edge neighbors 6 uncolored elements in $T(G)$, so the remaining lists have size 5, and Lemma 2.12 completes the coloring.

If $|V(C)| = 4$, then the remaining cases are that two consecutive uncolored halfedges have a common endpoint or that both pairs of opposite uncolored halfedges have a common endpoint. In the first case, we have found a 3-cycle, and we use that cycle as C instead, applying one of the cases above.

In the second case, let u and v be the two common neighbors for the pairs of halfedges (they are distinct, since otherwise consecutive halfedges have a common endpoint). If u and v are adjacent, then $V(C) \cup \{u, v\}$ induces $K_{3,3}$. After the initial phase, we erase the colors on u and v. With $H = K_{3,3}$, each uncolored vertex or edge neighbors 6 uncolored elements in $T(G)$, so the remaining lists have size 5, and Lemma 2.12 completes the coloring.

If u and v are not adjacent, then replace C with the cycle C' induced by $(V(C) \cup u) - \{w\}$, where w is a vertex of C not adjacent to u. Since $uv \notin E(G)$, we can choose one edge incident to each vertex of C' so that at most one pair of opposite incident edges has a common endpoint. This puts us in an earlier case, all of which have been resolved.

$\square$

2.3 List-coloring the Square of a Subcubic Graph

We study the problem of coloring the square of a graph. In this section, we only consider graphs with no loops and no multiple edges. Since each component of a graph can be colored independently, we also only consider connected graphs. The *square* of a graph G, denoted G^2, has the same vertex set as G and has an

edge between two vertices if the distance between them in G is at most 2. We use $\chi(G)$ to denote chromatic number G. We use $\Delta(G)$ to denote the largest degree in G. We say that a graph G is subcubic if $\Delta(G) \leq 3$.

Wegner [50] initiated the study of the chromatic number for squares of planar graphs. This topic has been actively studied lately due to his conjecture.

Conjecture. *(Wegner [50]) Let G be a planar graph. The chromatic number $\chi(G^2)$ of G^2 is at most 7 if $\Delta(G) = 3$, at most $\Delta(G) + 5$ if $4 \leq \Delta(G) \leq 7$, and at most $\lfloor \frac{3\Delta(G)}{2} \rfloor + 1$ otherwise.*

Thomassen [41] proved Wegner's conjecture for $\Delta(G) = 3$, but it is still open for all values of $\Delta(G) \geq 4$. The best known upper bounds are due to Molloy and Salavatipour [39]. Better results can be obtained for special classes of planar graphs. Borodin *et al.* [6] and Dvořák *et al.* [36] proved that $\chi(G^2) = \Delta(G) + 1$ if G is a planar graph G with sufficiently large maximum degree and girth at least 7. A natural strengthening of this problem is to study the list chromatic number of the square of a planar graph.

Kostochka and Woodall [34] conjectured that $\chi_l(G^2) = \chi(G^2)$ for every graph G. Motivated by this conjecture, we consider the problem of computing $\chi_l(G^2)$ when G is subcubic. If G is subcubic, then clearly $\Delta(G^2) \leq (\Delta(G))^2 \leq 9$. It is an easy exercise to show that the Petersen graph is the only subcubic graph G whose square is a complete graph. Therefore, by the list-coloring version of Brook's Theorem in [13], we conclude that if G is subcubic and G is not the Petersen graph, then $\chi_l(G^2) \leq \Delta(G^2) \leq 9$. In fact, we show that this upper bound can be strengthened. We say that a subcubic graph is *non-Petersen* if it is not the Petersen graph.

Theorem 2.15. *If G is a non-Petersen subcubic graph, then $\chi_l(G^2) \leq 8$.*
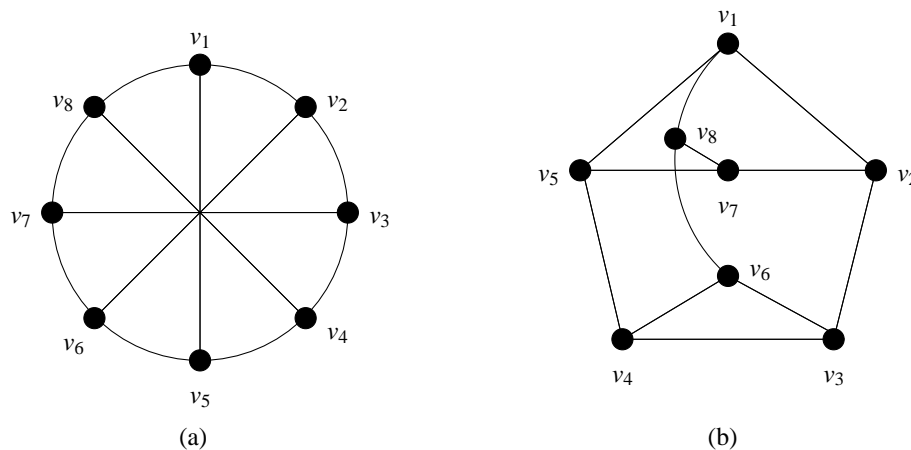
Figure 2.7. Two graphs, each on 8 vertices; each has K_8 as its square. (a) An 8-cycle $v_1, v_2, \ldots, v_8$ with "diagonals" (i.e. the additional edges are $v_i v_{i+4}$ for each $i \in \{1,2,3,4\}$). This graph has girth 4. (b) This graph has girth 3.

Theorem 2.15 is best possible, as illustrated by the graphs in Figure 3.2.2. The graph on the left has girth 4; the graph on the right has girth 3. The square of each graph is K_8. Thus, each graph has list-chromatic number 8. In fact, there are infinitely many non-Petersen subcubic graphs G such that $\chi_l(G^2) = 8$. Let H be

the Petersen graph with an edge removed. Note that $H^2 \supset K_8$. Hence, any graph G which contains H as a subgraph satisfies $\chi_l(G^2) \geq 8$.

Throughout this discussion, we use the idea of *saving a color* at a vertex v. By this we mean that we assign colors to two neighbors of v in G^2 but we only reduce the list of colors available at vertex v by one. A typical example of this occurs when v is adjacent to vertices v_1 and v_2 in G^2, v_1 is not adjacent to v_2 in G^2, and $|L(v_1)| + |L(v_2)| > |L(v)|$. This inequality implies that either $L(v_1)$ and $L(v_2)$ have a common color or that some color appears in $L(v_1) \cup L(v_2)$ but not in $L(v)$. In the first case, we save by using the same color on vertices v_1 and v_2. In the second case, we use a color in $(L(v_1) \cup L(v_2)) \setminus L(v)$ on the vertex where it appears and we color the other vertex arbitrarily.

We say that a graph G is *k-minimal* if G^2 is not k-choosable, but the square of every proper subgraph of G is k-choosable. A *configuration* in a graph G is an induced subgraph. We say that a configuration is *k-reducible* if it cannot appear in a k-minimal graph (we will be interested in the case $k = 8$).

2.3.1 Main results

We begin this section by proving several structural lemmas about 8-minimal subcubic graphs. We conclude by showing that if G is a non-Petersen subcubic graph, then $\chi_l(G^2) \leq 8$.

Lemma 2.16. *If G is a subcubic graph, then for any edge uv we have $\chi_l(G^2 \setminus \{u, v\}) \leq 8$.*

Proof: For every vertex w other than u and v, we define the *distance class* of w to be the distance in G from w to edge uv. We greedily color the vertices of $G^2 \setminus \{u, v\}$ in order of decreasing distance class. We claim that lists of size 8 suffice. Note that $|N(w)| \leq 9$ for every vertex w, which ensures that lists of size 10 suffices. If at least two vertices in $N(w)$ are uncolored when we color w, then having 8 colors in the list at w suffices.

Suppose that w has distance at least 2 from $\{u, v\}$. Let x and y be the first two vertices after w on a shortest path in G from w to $\{u, v\}$. Since vertices x and y are in lower distance classes than w, they are both uncolored when we color w, as desired. If $w \in N_G(u) \cup N_G(v)$, then u and v are uncolored when we color w. Again a list of size 8 suffices. $\qquad\square$

Lemma 2.16 shows that if G is a subcubic graph, then lists of size 8 are sufficient to color all but any two specified adjacent vertices of G^2. Hence, if H is any subgraph that contains an edge, then we can color $G^2 \setminus V(H)$ from lists of size 8. The next lemma relies on the same idea as Lemma 2.16 but applies in a more general context.

Given a graph G, a partial coloring of G^2, and an uncolored vertex v, we let $excess(v) = 1 + l(v) - m(v)$, where $l(v)$ is the number of colors available in the list at v after the partial coloring and $m(v)$ is the number of uncolored neighbors of v in G^2. Since $\Delta(G^2) \leq 9$ and we assign lists of size 8, always $excess(v) \geq 0$. Intuitively, $excess(v)$ measures how many colors we have "saved" on v; colors are saved either from using the same color on two neighbors of v or simply because v has fewer than 9 neighbors in G^2. For example, if two neighbors of v in G^2 receive the same color in the partial coloring, then $excess(v) \geq 1$. Similarly, if v lies on a 4-cycle or a 3-cycle, then $excess(v) \geq 1$ or $excess(v) \geq 2$, respectively. Vertices with positive excess play a special role in finishing a partial coloring.

Lemma 2.17. *Let G be a subcubic graph, and let L be a list assignment for G with lists of size 8. Suppose that G^2 has a partial coloring from L. Suppose also that vertices u and v are uncolored, are adjacent in G^2, and that $excess(u) \geq 1$ and $excess(v) \geq 2$. If we can order the uncolored vertices so that each vertex except u and v is followed somewhere by two adjacent vertices in G^2, then the partial coloring extends to an L-coloring of G^2.*

Proof: We will color the vertices greedily according to the order. Recall that for each vertex w, we have $|N(w)| \leq 9$. Since at least two vertices in $N(w)$ will be uncolored when we color w (for $w \notin \{u,v\}$), we will have a color available to use on w. Since u and v are the only vertices not succeeded by two adjacent vertices in G^2, they must be the last two vertices in the order. Because $excess(u) \geq 1$ and $excess(v) \geq 2$, we can finish the coloring by greedily coloring u and then v. $\qquad\square$

A simple but useful instance where Lemma 2.17 applies is when the uncolored vertices induce a connected subgraph and vertices u and v are adjacent and we can show that $excess(u) \geq 1$ and $excess(v) \geq 2$. In this case, for the needed ordering it suffices to order the vertices by decreasing distance (within the subgraph) from $\{u,v\}$. Whenever we say that we can "greedily finish a coloring", we will be using Lemma 2.17. Often, we will specify an order for the uncolored vertices; when we do not give an order it is because they induce a connected subgraph. The next two lemmas exhibit small configurations where we can apply Lemma 2.17.

Lemma 2.18. *If G is an 8-minimal subcubic graph, then G is 3-regular.*

Proof: If u is a vertex with $d(u) \leq 2$, and v be a neighbor of u, then $excess(v) \geq 1$ and $excess(u) \geq 3$. By Lemmas 2.16 and 2.17, $\chi_l(G^2) \leq 8$. $\qquad\square$

Lemma 2.19. *If G is an 8-minimal subcubic graph, then G has girth at least 4.*

Proof: The vertices of a 3-cycle in G have excess at least 2. By Lemmas 2.16 and 2.17, $\chi_l(G^2) \leq 8$. $\qquad\square$

Lemma 2.20. *If G is an 8-minimal subcubic graph, then G has girth at least 5.*

Proof: Suppose that G is an 3-minimal subcubic graph having a 4-cycle, and let L be an 8-uniform list assignment. Any vertex on a 4-cycle has excess at least 1. If v lies on two 4-cycles, then $excess(v) \geq 2$; if u is a neighbor of v on a 4-cycle, then Lemmas 2.16 and 2.17 apply. Therefore, we may assume that no vertex lies on two 4-cycles.

Let C be a 4-cycle in G. Label the vertices of C as v_1, v_2, v_3, v_4. Recall that G is 3-regular, by Lemma 2.18. Let u_i be the neighbor of v_i not on C. We may assume that these neighbors are distinct, since otherwise either G contains a 3-cycle or some vertex lies on two 4-cycles. Using Lemma 2.16, we choose colors for all vertices except those on the 4-cycle and their neighbors; call this coloring c. Let $L'(x)$ denote the list of remaining colors available at each uncolored vertex x.

Case 1: $d(u_1, v_3) = 3$. Note that $|L'(v_i)| \geq 6$ and $|L'(u_i)| \geq 2$. We assume that equality holds for v_1 (otherwise we throw away colors until it does), although not necessarily for the u_i; for example, if $d(u_1, u_2) = 2$, then $|L'(u_1)| \geq 3$ and $|L'(u_2)| \geq 3$. Since $|L'(u_1)| + |L'(v_3)| > |L'(v_1)|$, we can choose color c_1 for u_1 and

color c_2 for v_3 so that $|L'(v_1) \setminus \{c_1, c_2\}| \geq 5$. Since $\mathrm{excess}(v_2) \geq 1$ and $\mathrm{excess}(v_1) \geq 2$, we can finish the coloring by Lemma 2.17 (coloring greedily in the order $u_2, u_3, u_4, v_4, v_2, v_1$).

Case 2: $d(u_1, v_3) < 3$. Vertices u_1 and u_3 must be adjacent; by symmetry u_2 and u_4 must be adjacent. Now since u_1 and u_3 are adjacent and u_2 and u_4 are adjacent, we have $|L'(v_i)| \geq 7$ and $|L'(u_i)| \geq 4$ (we assume that equality holds for the v_is). Suppose that $d(u_1, u_2) = 3$. Since $|L'(u_1)| + |L'(u_2)| \geq 4 + 4 > 7 = |L'(v_1)|$, we can choose color c_1 for u_1 and color c_2 for u_2 such that $|L'(v_1) \setminus \{c_1, c_2\}| \geq 6$. Since $\mathrm{excess}(v_1) \geq 2$ and $\mathrm{excess}(v_2) \geq 1$, we can finish the coloring. Hence, we can assume that $d(u_1, u_2) < 3$.
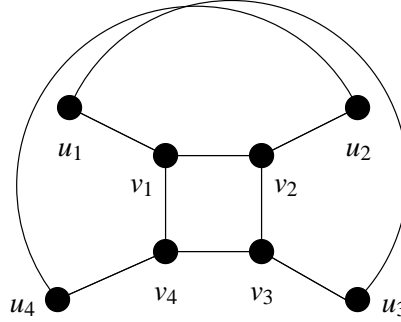
Figure 2.8. A 4-cycle with vertices v_1, v_2, v_3, v_4 and the adjacent vertices not on the 4-cycle: u_1, u_2, u_3, u_4, respectively. In Case 2 of Lemma 7, we also assume that vertices u_1 and u_3 are adjacent and that vertices u_2 and u_4 are adjacent.

Observe that u_1 and u_2 cannot be adjacent, since then v_1 lies on two 4-cycles. Thus, u_1 and u_2 must have a common neighbor. By symmetry, we can assume that u_1 and u_4 have a common neighbor. Since $d(u_1) = 3$ (and we have already accounted for two edges incident to u_1), vertices u_1, u_2, and u_4 must have a common neighbor x. However, then u_2, u_4, and x form a 3-cycle. By Lemma 2.19, this is a contradiction. $\square$

Lemma 2.21. *If G is an non-Petersen 8-minimal subcubic graph, then G does not contain two 5-cycles that share three consecutive vertices.*

Proof: Suppose G is a counterexample. Taken together, the two given 5-cycles form a 6-cycle, with one additional vertex adjacent to two vertices of the 6-cycle. Label the vertices of the 6-cycle $v_1, v_2, \ldots, v_6$ and label the final vertex v_7. Let v_7 be adjacent to v_1 and v_4. We consider three cases, depending on how many pairs of vertices on the 6-cycle are distance 3 apart. By Lemma 2.16, we color all vertices of G^2 except the 7 v_is.

Case 1: Both $d(v_2, v_5) \geq 3$ and $d(v_3, v_6) \geq 3$. Let $L'(v)$ denote the list of remaining colors available at each uncolored vertex v. In this case, $|L'(v_1)| \geq 5$, $|L'(v_4)| \geq 5$, $|L'(v_7)| \geq 5$ and $|L'(v_2)| \geq 4$, $|L'(v_3)| \geq 4$, $|L'(v_5)| \geq 4$, $|L'(v_6)| \geq 4$. We assume equality holds. We consider two subcases.

Subcase 1.1: $L'(v_2) \cap L'(v_5) \neq \emptyset$ or $L'(v_3) \cap L'(v_6) \neq \emptyset$. Without loss of generality, we may assume that $L'(v_2) \cap L'(v_5) \neq \emptyset$. Color v_2 and v_5 with some color $c_1 \in L'(v_2) \cap L'(v_5)$. Since $|L'(v_3) \setminus \{c_1\}| + |L'(v_6) \setminus \{c_1\}| > |L'(v_7) \setminus \{c_1\}|$, we can choose color c_2 for v_3 and color c_3 for v_6 such that $|L'(v_7) \setminus \{c_1, c_2, c_3\}| \geq 3$. Greedily color the remaining vertices in the order v_1, v_4, v_7.

Subcase 1.2: $L'(v_2) \cap L'(v_5) = \emptyset$ and $L'(v_3) \cap L'(v_6) = \emptyset$. Color v_1, v_4, v_7 so that no two vertices among v_2, v_3, v_5, v_6 have only one available color remaining. Call these new lists $L''(v)$. Note that $|L''(v_2)| + |L''(v_5)| \geq 5$ and $|L''(v_3)| + |L''(v_6)| \geq 5$. Hence we can color v_2, v_3, v_5, v_6.
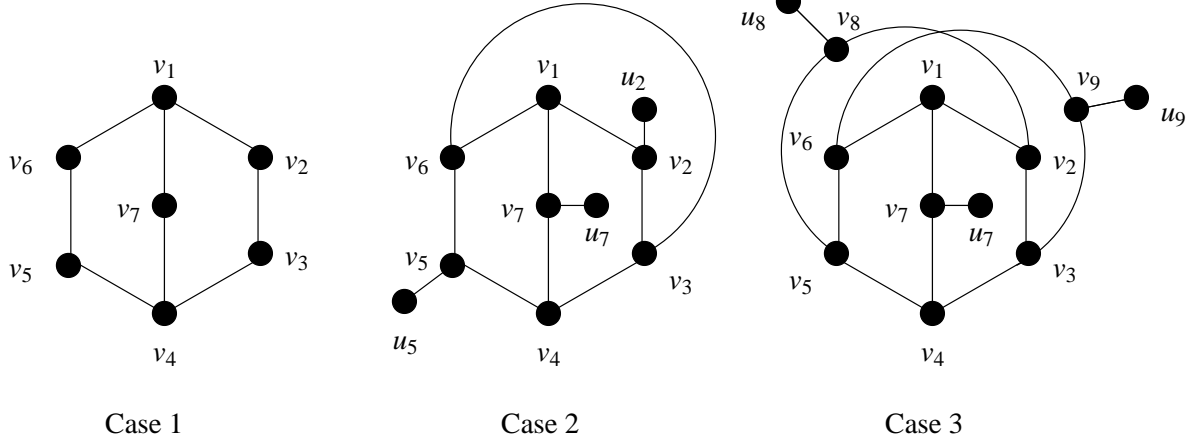
Figure 2.9. Lemma 8 considers two 5-cycles that share two consecutive edges. In Cases 2 and 3 of Lemma 8, we consider additional adjacencies.

Case 2: Exactly one of $d(v_2, v_5)$ or $d(v_3, v_6)$ is 2. Without loss of generality, we may assume that $d(v_2, v_5) \geq 3$ and $d(v_3, v_6) = 2$. Recall from Lemma 2.18 that G is 3-regular. Let u_2, u_5, and u_7 be the vertices not yet named that are adjacent to v_2, v_5, and v_7, respectively. We cannot have $u_2 = u_5$, since we have $d(v_2, v_5) \geq 3$. Note that $d(u_2, v_4) \geq 3$ unless $u_2 = u_7$. Similarly, $d(u_5, v_1) \geq 3$ unless $u_5 = u_7$. Moreover, we cannot have $u_2 = u_7$ or $u_5 = u_7$, since this forms a 4-cycle. Hence, $d(u_2, v_4) = 3$ and $d(u_5, v_1) = 3$. Uncolor vertex u_2. Let $L'(v)$ denote the list of remaining available colors at each vertex v. We have $|L'(v_1)| \geq 6$, $|L'(v_2)| \geq 5$, $|L'(v_3)| \geq 6$, $|L'(v_4)| \geq 5$, $|L'(v_5)| \geq 4$, $|L'(v_6)| \geq 5$, $|L'(v_7)| \geq 5$, and $|L'(u_2)| \geq 2$. We assume that equality holds. We consider two subcases.

Subcase 2.1: $L'(u_2) \cap L'(v_4) \neq \emptyset$. Color u_2 and v_4 with some color $c_1 \in L'(u_2) \cap L'(v_4)$. Now choose color c_2 for v_2 and color c_3 for v_5 such that $|L'(v_3) \setminus \{c_1, c_2, c_3\}| \geq 4$. Let $L''(v) = L'(v) \setminus \{c_1, c_2, c_3\}$. The new lists satisfy $|L''(v_1)| \geq 3, |L''(v_3)| \geq 4, |L''(v_6)| \geq 2, |L''(v_7)| \geq 2$. Greedily color the remaining vertices in the order v_7, v_6, v_1, v_3.

Subcase 2.2: $L'(u_2) \cap L'(v_4) = \emptyset$. We have two subcases here. If $L'(v_2) \cap L'(v_5) \neq \emptyset$, then color v_2 and v_5 with a common color, and then color u_2 and v_4 to save a color at v_3. Now color the remaining vertices as in Subcase 2.1. If $L'(v_2) \cap L'(v_5) = \emptyset$, then color u_2 and v_4 to save a color at v_3. Now choose colors for v_6 and for v_7 such that vertices v_2 and v_5 each have at least one remaining color. Let $L''(v)$ denote the list of remaining available colors at each vertex v. Note that $|L''(v_1)| \geq 2, |L''(v_3)| \geq 3$, and $|L''(v_2)| + |L''(v_5)| \geq 5$ since $L'(v_2) \cap L'(v_5) = \emptyset$. In each case, we can color v_1, v_2, v_3, v_5.

Case 3: Both $d(v_2, v_5)$ and $d(v_3, v_6)$ are 2. Then v_2 and v_5 have a common neighbor, say v_8, and v_3 and v_6 have a common neighbor, say v_9. Let u_7, u_8, and u_9 be the third vertices adjacent to v_7, v_8, and v_9, respectively. We show that either $d(v_7, v_8) = 3$ or $d(v_7, v_9) = 3$ or $d(v_8, v_9) = 3$. Note that $d(v_7, v_8) = 3$ unless $u_7 = u_8$. Similarly, $d(v_7, v_9) = 3$ unless $u_7 = u_9$ and $d(v_8, v_9) = 3$ unless $u_8 = u_9$. However, we cannot have $u_7 = u_8 = u_9$, since G is not the Petersen graph. Hence, by symmetry, assume that $u_7 \neq u_8$. So $d(v_7, v_8) = 3$. In this case, consider the two 5-cycles: $v_1 v_2 v_3 v_4 v_7 v_1$ and $v_2 v_3 v_4 v_5 v_8 v_2$; they share three consecutive vertices such that when labeled as above $d(v_2, v_5) = 3$. Hence, the graph can be handled as in case 1 or 2. $\qquad \square$

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Lemma 2.22. *If G is an non-Petersen 8-minimal subcubic graph, then G does not contain two 5-cycles that share an edge.*

Proof: Suppose G is a counterexample. By Lemmas 2.18-2.20, we know that G is 3-regular and that $g(G) \geq 5$. Taken together, these 5-cycles form an 8-cycle, with a chord. Label the vertices of the 8-cycle $v_1, v_2, \ldots, v_8$ with an edge between v_1 and v_5. By Lemmas 2.20 and 2.21, we know that $d(v_2, v_6) = 3$. Similarly, we know that $d(v_4, v_8) = 3$. By Lemma 2.16, we color all vertices of G^2 except the 8 v_is. Let $L'(v)$ denote the list of remaining available colors at each vertex v. Note that $|L'(v_1)| \geq 6$, $|L'(v_2)| \geq 4$, $|L'(v_3)| \geq 3$, $|L'(v_4)| \geq 4$, $|L'(v_5)| \geq 6$, $|L'(v_6)| \geq 4$, $|L'(v_7)| \geq 3$, and $|L'(v_8)| \geq 4$. We assume that equality holds.

Case 1: There exists a color $c_1 \in L'(v_4) \cap L'(v_8)$. Use color c_1 on v_4 and v_8. Since $|L'(v_2) \setminus \{c_1\}| + |L'(v_6) \setminus \{c_1\}| > |L'(v_5) \setminus \{c_1\}|$, we can choose color c_2 for v_2 and color c_3 for v_6 such that $|L'(v_5) \setminus \{c_1, c_2, c_3\}| \geq 4$. Now since $\text{excess}(v_1) \geq 1$ and $\text{excess}(v_5) \geq 2$, we can finish the coloring by Lemma 2.17.
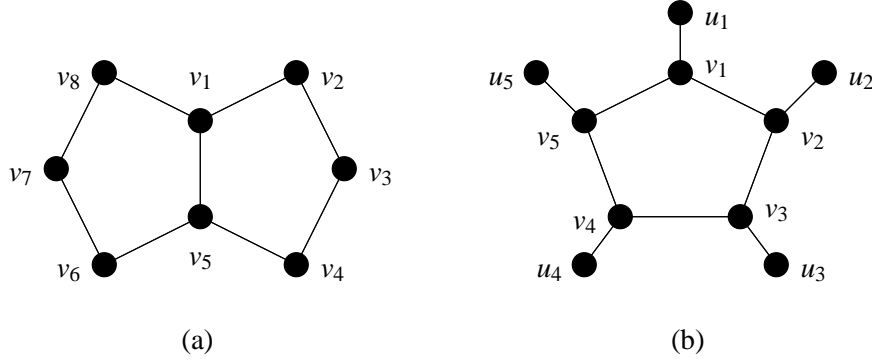
Figure 2.10. (a) Lemma 9 considers two 5-cycles that share an edges. (b) Lemma 10 considers a single 5-cycle.

Case 2: $L'(v_4) \cap L'(v_8) = \emptyset$. We can choose color c_1 for v_2 and color c_2 for v_6 such that $|L'(v_5) \setminus \{c_1, c_2\}| \geq 5$. Note that now $\text{excess}(v_5) \geq 1$. Now color v_3 and v_7 arbitrarily with colors from their lists; call them c_3 and c_4, respectively. Since $L'(v_4) \cap L'(v_8) = \emptyset$, the remaining lists for v_4 and v_8 have sizes summing to at least 4; call these lists $L''(v_4)$ and $L''(v_8)$. If $|L''(v_4)| \geq 3$, then $\text{excess}(v_4) \geq |L''(v_4)| - 1 = 2$, so by Lemma 2.17 we can finish the coloring. Similarly, if $|L''(v_8)| \geq 3$, then $\text{excess}(v_8) \geq |L''(v_8)| - 1 = 2$, so by Lemma 2.17 we can finish the coloring. So assume that $|L''(v_4)| = |L''(v_8)| = 2$. Arbitrarily color v_1 from its list; call the color c_3. Since $L'(v_4) \cap L'(v_8) = \emptyset$, either $|L''(v_4) \setminus \{c_3\}| = 2$ or $|L''(v_8 \setminus \{c_3\}| = 2$. In the first case, $\text{excess}(v_4) \geq 2$; in the second case, $\text{excess}(v_8) \geq 2$. In either case, we can greedily finish the coloring by Lemma 2.17. $\qquad\square$

Lemma 2.23. *If G is an non-Petersen 8-minimal subcubic graph, then $g(G) > 5$.*

Proof: Suppose G is a counterexample. By Lemmas 2.18-2.20, we know that G is 3-regular and that $g(G) = 5$. Let $v_1 v_2 v_3 v_4 v_5 v_1$ be a 5-cycle and let u_i be the neighbor of vertex v_i not on the 5-cycle.

By Lemma 2.16, we can greedily color all vertices except the u_is and v_is. Let $L'(v)$ denote the list of remaining available colors at each vertex v. Note that $|L'(u_i)| \geq 2$ and $|L'(v_i)| \geq 6$. We assume that equality

holds for the v_is. By Lemma 2.21, we know that $d(u_i, v_{i+2}) = d(u_i, v_{i+3}) = 3$ for all i (subscripts are modulo 5). By Lemma 2.22 we also know that $d(u_i, u_{i+1}) = 3$.

Case 1: There exists a color $c_1 \in L'(u_1) \cap L'(v_3)$. Use c_1 on u_1 and v_3. Greedily color vertices u_2, u_3, u_4; call these colors c_2, c_3, c_4, respectively. Now $|L'(v_1) \setminus \{c_1, c_2\}| = 4$, $|L'(v_2) \setminus \{c_1, c_2, c_3\}| \geq 3$, and $|L'(u_5)| \geq 2$. We can choose color c_5 for u_5 and color c_6 for v_2 such that $|L'(v_1) \setminus \{c_1, c_2, c_5, c_6\}| \geq 3$. Now greedily color the remaining vertices in the order v_4, v_5, v_1.

Case 2: There exists a color $c_1 \in L'(u_1) \cap L'(u_2)$. Use color c_1 on u_1 and u_2. Now $|L'(v_5) \setminus \{c_1\}| + |L'(u_3)| > |L'(v_2) \setminus \{c_1\}|$, so we can choose color c_2 for v_5 and color c_3 for v_3 so that excess$(v_2) \geq 2$. Note that excess$(v_1) \geq 1$. Hence, after we greedily color u_5, we can extend the partial coloring to the remaining uncolored vertices by Lemma 2.17.

Case 3: $L'(u_i) \cap L'(u_{i+1}) = \emptyset$ and $L'(u_i) \cap L'(v_{i+2}) = \emptyset$ for all i. By symmetry, we can assume $L'(u_i) \cap L'(v_{i+3}) = \emptyset$ for all i. We now show that we can color each vertex with a distinct color. Suppose not.

By Hall's Theorem [52], there exists a subset of the uncolored vertices V_1 such that $|\cup_{v \in V_1} L'(v)| < |V_1|$. Recall that $|L'(u_i)| \geq 2$ and $|L'(v_i)| = 6$ for all i. Clearly, $2 < |V_1| \leq 10$. If $|V_1| \leq 6$, then $V_1 \subseteq \{u_1, u_2, u_3, u_4, u_5\}$. Any three u_is contain a pair u_j, u_{j+1}; their lists are disjoint, so $|\cup_{v \in V_1} L'(v)| \geq |L'(u_j)| + |L'(u_{j+1})| \geq 4$. If $|V_1| = 5$, then $V_1 = \{u_1, u_2, u_3, u_4, u_5\}$. However, each color appears on at most two u_is, hence $|\cup_{v \in V_1} L'(v)| \geq 10/2 = 5$. So say $|V_1| \geq 7$. The Pigeonhole principle implies that V_1 must contain a pair u_i, v_{i+2}. Since lists $L'(u_i)$ and $L'(v_{i+2})$ are disjoint, we have $|\cup_{v \in V_1} L'(v)| \geq |L'(u_i)| + |L'(v_{i+2})| = 2 + 6 = 8$. Hence, $|V_1| \geq 9$. Now V_1 must contain a triple u_i, u_{i+1}, v_{i+3}. Since their lists are pairwise disjoint, we get $|\cup_{v \in V_1} L'(v)| \geq |L'(u_i)| + |L'(u_{i+1})| + |L'(v_{i+3})| = 2 + 2 + 6 = 10$. This is a contradiction. Thus, we can finish the coloring. $\square$

Now we prove that if G is 8-minimal, then G does not contain a 6-cycle.

Lemma 2.24. *If G is an non-Petersen 8-minimal subcubic graph, then $g(G) > 6$.*

Proof: Let G be a counterexample. By Lemma 2.23, we know that $g(G) > 5$. Hence, a counterexample must have girth 6. We show how to color G from lists of size 8. First, we prove that if $H = C_6$, then $\chi_l(H^2) = 3$. Our plan is to first color all vertices except those on the 6-cycle, then color the vertices of the 6-cycle.

Claim: If $H = C_6$, then $\chi_l(H^2) = 3$.

Label the vertices $v_1, v_2, v_3, v_4, v_5, v_6$ in succession. Let $L'(v)$ denote the list of available colors at each vertex v. We consider separately the cases where $L'(v_1) \cap L'(v_4) \neq \emptyset$ and where $L'(v_1) \cap L'(v_4) = \emptyset$.

Case 1: There exists a color $c_1 \in L'(v_1) \cap L'(v_4)$. Use color c_1 on v_1 and v_4. Note that $|L'(v_i) \setminus \{c_1\}| \geq 2$ for each $i \in \{2, 3, 5, 6\}$. If there exists a color $c_2 \in (L'(v_2) \cap L'(v_5)) \setminus \{c_1\}$, then use color c_2 on v_2 and v_5. Now greedily color v_3 and v_6. So suppose there is no color in $(L'(v_2) \cap L'(v_5)) \setminus \{c_1\}$. Color v_3 arbitrarily; call it color c_3. Either $|L'(v_2) \setminus \{c_1, c_3\}| \geq 2$ or $|L'(v_5) \setminus \{c_1, c_3\}| \geq 2$. In the first case, greedily color v_5, v_6, v_2. In the second case, greedily color v_2, v_6, v_5.

Case 2: $L'(v_1) \cap L'(v_4) = \emptyset$. By symmetry, we assume $L'(v_2) \cap L'(v_5) = \emptyset$ and $L'(v_3) \cap L'(v_6) = \emptyset$. Color v_1 arbitrarily; call it color c_1. If there exists i such that $|L'(v_i) \setminus \{c_1\}| = 2$, then color v_4 from $c_2 \in L'(v_4) \setminus L'(v_i)$; otherwise color v_4 arbitrarily. Let $L''(v_j) = L'(v_j) \setminus \{c_1, c_2\}$ for all $j \in \{2, 3, 5, 6\}$. Note that $|L''(v_2)| +$

$|L''(v_5)| \geq 4$ and $|L''(v_3)| + |L''(v_6)| \geq 4$. Also, note that there is at most one k in $\{2,3,5,6\}$ such that $|L''(k)| = 1$. So by symmetry we consider two subcases.

Subcase 2.1: $|L''(v_j)| \geq 2$ for every $j \in \{2,3,5,6\}$. We can finish as in case 1 above.

Subcase 2.2: $|L''(v_2)| = 1$, $|L''(v_3)| \geq 2$, $|L''(v_6)| \geq 2$, and $|L''(v_5)| \geq 3$. We color greedily in the order v_2, v_3, v_6, v_5.

This finishes the proof of the claim; now we prove the lemma.

Let u and v be adjacent vertices on a 6-cycle C. By Lemma 2.16, color all vertices except the vertices of C. Since $g(G) = 6$, C has no chords. Similarly, no two vertices of C have a common neighbor not on C. Note that each vertex of C has at least three available colors. Hence, by the Claim we can finish the coloring. $\square$

The fact that $\chi_l(C_6^2) = 3$ is a special case of a theorem by Juvan, Mohar, and Škrekovski [32]. They showed that for any k, if $G = C_{6k}$, then $\chi_l(G^2) = 3$. Their proof uses algebraic methods and is not constructive. This fact is also a special case of a result by Fleischner and Steibitz [16]; their result also relies on algebraic methods.

Lemma 2.25. *Let C be a shortest cycle in an non-Petersen 8-minimal subcubic graph G. If u_1 and u_2 are each distance 1 from C, then u_1 and u_2 are nonadjacent.*

Proof: Let C be a shortest cycle in G. Lemma 2.24 implies that $|V(C)| \geq 7$. Let $v_1, v_2, \ldots, v_k$ be the vertices of C. Recall that G is 3-regular. Let u_i be the neighbor of v_i that is not on C. Suppose that there exists u_i adjacent to u_j. Let d be the distance from v_i to v_j *along C*. By combining the path $v_i u_i u_j v_j$ with the shortest path along C from v_i to v_j, we get a cycle of length $3 + d \leq 3 + \lfloor |V(C)|/2 \rfloor < |V(C)|$. This contradicts the fact that C is a shortest cycle in G. $\square$
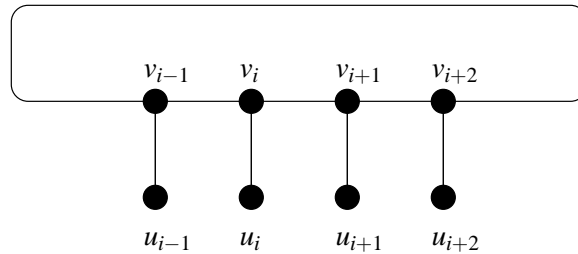
Figure 2.11. In the proof of Theorem 1, we frequently consider four consecutive vertices on a cycle and their neighbors off the cycle.

We are now ready to prove Theorem 2.15.

Theorem 2.15. *If G is an non-Petersen subcubic graph, then $\chi_l(G^2) \leq 8$.*

Proof: Let G be a counterexample. By Lemma 2.18, we know that G is 3-regular. By Lemma 2.24, we know that G has girth at least 7. Let C be a shortest cycle in G. Let $v_1, v_2, \ldots, v_k$ be the vertices of C. Let u_i be the neighbor of v_i that is not on C. Let H be the union of the v_is and the u_is. By Lemma 2.16, we can color $G^2 \setminus V(H)$. Let $L'(v)$ denote the list of available colors at each vertex v. Note that $|L'(v_i)| \geq 6$ and $|L'(u_i)| \geq 2$ for all i. We assume that equality holds.

Claim 1: If we can choose color c_1 for u_i and color c_2 for u_{i+1} such that $|L'(v_i) \setminus \{c_1, c_2\}| \geq 5$ and $|L'(v_{i+1}) \setminus \{c_1, c_2\}| \geq 5$, then we can extend the coloring to all of G^2.

Use colors c_1 and c_2 on u_i and u_{i+1}. Since $|L'(u_{i-1})| = 2$ and $|L'(v_{i+2}) \setminus \{c_2\}| \geq 5$ and $|L'(v_i) \setminus \{c_1, c_2\}| \geq 5$, we can choose color c_3 for u_{i-1} and color c_4 for v_{i+2} so that $|L'(v_i) \setminus \{c_1, c_2, c_3, c_4\}| \geq 4$. Color u_{i+2} arbitrarily. Now since $\text{excess}(v_{i+1}) \geq 1$ and $\text{excess}(v_i) \geq 2$, we can greedily finish the coloring by Lemma 2.17.

Claim 2: If we can choose color c_1 for u_i such that $|L'(v_i) \setminus \{c_1\}| = 6$, then we can extend the coloring to all of G^2.

Use color c_1 on u_i. Since $|L'(u_{i-1})| = 2$ and $|L'(v_{i+1}) \setminus \{c_1\}| \geq 5$ and $|L'(v_{i-1}) \setminus \{c_1\}| \geq 5$, we can chose color c_2 for u_{i-1} and color c_3 for v_{i+1} such that $|L'(v_{i-1}) \setminus \{c_1, c_2, c_3\}| \geq 4$. If $c_2 = c_3$, then we use c_2 on vertices u_{i-1} and v_{i+1}; Now $\text{excess}(v_{i-1}) \geq 1$ and $\text{excess}(v_i) \geq 2$. So after we greedily color u_{i+1}, we can finish by Lemma 2.17. Hence, we can assume $c_2 \neq c_3$. Note that either $c_2 \notin L'(v_{i-1})$ or $c_3 \notin L'(v_{i-i})$. If $c_2 \notin L'(v_{i-1})$, then use c_2 on u_{i-1}; now we can finish by Claim 1. Hence, we can assume $c_3 \notin L'(v_{i-1})$. Use c_3 on v_{i+1}, but don't color u_{i-1}. Greedily color u_{i+1} and u_{i+2}; call these colors c_4 and c_5, respectively. We may assume that $|L'(v_i) \setminus \{c_1, c_3, c_4\}| = 4$ (otherwise, we can finish greedily as above). We also know that $|L'(u_{i-1})| = 2$ and $|L'(v_{i+2}) \setminus \{c_3, c_4, c_5\}| \geq 3$. Hence, we can choose color c_6 for u_{i-1} and color c_7 for v_{i+2} such that $|L'(v_i) \setminus \{c_1, c_3, c_4, c_6, c_7\}| \geq 3$. Now since $\text{excess}(v_{i-1}) \geq 1$ and $\text{excess}(v_i) \geq 2$, we can finish by Lemma 2.17.

Claim 3: If we can choose color c_1 for u_{i+1} such that $|L'(v_i) \setminus \{c_1\}| = 6$, then we can extend the coloring to all of G^2.

Use color c_1 on u_{i+1}. Since $|L'(u_i)| = 2$ and $|L'(v_{i+2}) \setminus \{c_1\}| \geq 5$ and $|L'(v_{i+1}) \setminus \{c_1\}| \geq 5$, we can choose color c_2 for u_i and color c_3 for v_{i+2} such that $|L'(v_{i+1}) \setminus \{c_1, c_2, c_3\}| \geq 4$. Now we are in the same situation as in the proof of Claim 2. If $c_2 = c_3$, then we use color c_2 on u_i and v_{i+2} and color greedily as in Claim 2. If $c_2 \notin L'(v_{i+1}) \setminus \{c_1\}$, then we use c_2 on u_i and we can finish by Claim 1. Hence we must have $c_3 \notin L'(v_{i+1})$. Use c_3 on $L'(v_{i+2})$. As in Claim 2, we have $|L'(v_i) \setminus \{c_1, c_3\}| \geq 5$ and $|L'(v_{i+1}) \setminus \{c_1, c_3\}| \geq 5$. Hence, we can finish as in Claim 2.

Remark: Claim 2 and Claim 3 imply that for every i we have $L'(u_{i-1}) \cup L'(u_i) \cup L'(u_{i+1}) \subseteq L'(v_i)$. Furthermore, Claim 1 shows that $L'(u_i) \cap L'(u_{i+1}) = \emptyset$ for all i. To show that $L'(u_{i-1})$, $L'(u_i)$, and $L'(u_{i+1})$ are pairwise disjoint we prove Claim 4.

Claim 4: If we can choose color c_1 for u_{i-1} and color c_2 for u_{i+1} such that $|L'(v_i) \setminus \{c_1, c_2\}| \geq 5$, then we can extend the coloring to G^2.

Use color c_1 on u_{i-1} and color c_2 and u_{i+1}. Since $|L'(u_i)| = 2$ and $|L'(v_{i+2}) \setminus \{c_2\}| \geq 5$ and $|L'(v_{i+1}) \setminus \{c_2\}| \geq 5$, we can choose color c_3 for u_i and color c_4 for v_{i+2} such that $|L'(v_{i+1}) \setminus \{c_2, c_3, c_4\}| \geq 4$. If $c_3 = c_4$, then we use color c_3 on u_i and v_{i+2}; since $\text{excess}(v_{i+1}) \geq 1$ and $\text{excess}(v_i) \geq 2$, we can finish by Lemma 2.17. So either $c_3 \notin L'(v_{i+1})$ or $c_4 \notin L'(v_{i+1})$.

Suppose $c_3 \notin L'(v_{i+1})$. Use c_3 on u_i. Since $|L'(v_{i-1}) \setminus \{c_1, c_3\}| \geq 4$ and $|L'(u_{i+2})| = 2$ and $|L'(v_{i+1}) \setminus \{c_3\}| \geq 5$, we can choose color c_5 for v_{i-1} and color c_6 for u_{i+2} such that $|L'(v_{i+1}) \setminus \{c_2, c_3, c_5, c_6\}| \geq 4$. Now since $\text{excess}(v_i) \geq 1$ and $\text{excess}(v_{i+1}) \geq 2$, we can finish by Lemma 2.17.

Suppose instead that $c_4 \notin L'(v_{i+1})$. Use c_4 on v_{i+2}. Color u_{i+2} and u_{i+3} arbitrarily; call these colors c_5 and c_6, respectively. Since $|L'(u_i)| = 2$ and $|L'(v_{i+3}) \setminus \{c_4, c_5, c_6\}| \geq 3$ and $|L'(v_{i+1}) \setminus \{c_2, c_4, c_5\}| = 4$, we can choose color c_7 for u_i and color c_8 for v_{i+3} such that $|L'(v_{i+1}) \setminus \{c_2, c_4, c_5, c_7, c_8\}| \geq 3$. Now since $\text{excess}(v_i) \geq 1$ and $\text{excess}(v_{i+1}) \geq 2$, we can finish by Lemma 2.17.

Claim 5: We can extend the coloring to G^2 in the following way. Color each u_j arbitrarily; let $c(u_j)$ denote the color we use on each u_j. Now assign a color to each v_j from $L'(u_j) \setminus \{c(u_j)\}$.

For each j, Claim 4 implies that $L'(u_{j-1})$, $L'(u_j)$, and $L'(u_{j+1})$ are pairwise disjoint. Hence, each v_j receives a color not in $\{c(u_{j-1}), c(u_j), c(u_{j+1})\}$. Similarly, since $L'(u_j)$ is disjoint from $L'(u_{j-2}), L'(u_{j-1}), L'(u_{j+1})$, and $L'(u_{j+2})$, vertex v_j receives a color not in $\{c(v_{j-2}), c(v_{j-1}), c(v_{j+1}), c(v_{j+2})\}$. Hence, the coloring of G^2 is valid. □

2.3.2 Efficient Algorithms

Since the proof of Theorem 2.15 colors all but a constant number of vertices greedily, it is not surprising that the algorithm can be made to run in linear time. For completeness, we give the details.

If G is not 3-regular or G has girth at most 6, then we find a small subgraph H (one listed in Lemmas 2.18-2.24) that contains a low degree vertex or a shortest cycle. It is easy to greedily color $G^2 \setminus V(H)$ in linear time (for example, using breadth-first search). Since H has constant size, we can finish the coloring in constant time.

Say instead that G is 3-regular and has girth at least 7. Choose an arbitrary vertex v. Find a shortest cycle through v (for example, using breadth-first search); call it C. Let H be C and vertices at distance 1 from C. We greedily color $G^2 \setminus V(H)$ in linear time. Using the details given in the proof of Theorem 2.3.1, we can finish the coloring in time linear in the size of H.

2.3.3 Future Work

As we mentioned in the introduction, Theorem 2.15 is best possible, since there are infinitely many non-Petersen subcubic graphs G such that $\chi_l(G^2) = 8$ (for example, any graph which contains the Petersen graph with one edge removed). However, it is natural to ask whether the result can be extended to graphs with arbitrary maximum degree. Let G be a graph with maximum degree $\Delta(G) = k$. Since $\Delta(G^2) \leq k^2$, we immediately get that $\chi_l(G^2) \leq k^2 + 1$. If $G^2 \neq K_{k^2+1}$, then by the list-coloring version of Brook's Theorem [13], we have $\chi_l(G^2) \leq k^2$. Hoffman and Singleton [26] made a thorough study of graphs G with maximum degree k such that $G^2 = K_{k^2+1}$. They called these *Moore Graphs*. They showed that a unique Moore Graph exists when $\Delta(G) \in \{2, 3, 7\}$ and possibly when $\Delta(G) = 57$ (which is unknown), but that no Moore Graphs exist for any other value of $\Delta(G)$. (When $\Delta(G) = 3$, the unique Moore Graph is the Petersen Graph). Hence, if $\Delta(G) \notin \{2, 3, 7, 57\}$, we know that $\chi_l(G^2) \leq \Delta(G)^2$. As in Theorem 2.15, we believe that we can improve this upper bound.

Conjecture 2.26. *If G is a graph with maximum degree $\Delta(G) = k$ and G is not a Moore Graph, then $\chi_l(G^2) \leq k^2 - 1$.*

Erdős, Fajtlowitcz and Hoffman [14] considered graphs G with maximum degree k such that $G^2 = K_{k^2}$. The proved the following result, which provides evidence in support of our conjecture.

Theorem. *(Erdős, Fajtlowitcz and Hoffman [14]) Apart from the cycle C_4, there is no graph G with maximum degree k such that $G^2 = K_{k^2}$.*

We extend this result to give a bound on the clique number $\omega(G^2)$ of the square of a non-Moore graph G with maximum degree k.

Lemma 2.27. *If G is not a Moore graph and G has maximum degree $k \geq 3$, then G^2 has clique number* $\omega(G^2) \leq k^2 - 1$.

Proof: If G is a counterexample, then by the Theorem of Erdős, Fajtlowitcz and Hoffman, we know that G^2 properly contains a copy of K_{k^2}. Choose adjacent vertices u and v_1 such that v_1 is in a clique of size k^2 (in G^2) and u is not in that clique; call the clique H. Note that $|N[v_1]| \leq k^2 + 1$, so all vertices in $N[v_1]$ other than u must be in H. Label the neighbors of u as v_is. Note that no v_i is on a 4-cycle. If so, then $|N[v_i]| \leq k^2$; since $u \in N[v_i]$ and $u \notin V(H)$, we get $|V(H)| \leq k^2 - 1$, which is a contradiction.

Note that each neighbor of a vertex v_i (other than u) must be in H. Since no v_i lies on a 4-cycle, each pair v_i, v_j have u as their only common neighbor. So the v_is and their neighbors (other than u) are k^2 vertices in H. But u is within distance 2 of each of these k^2 vertices in H. Hence, adding u to H yields a clique of size $k^2 + 1$. This is a contradiction. $\qquad\square$

We believe that Conjecture 2.26 can probably be proved using an argument similar to our proof of Theorem 2.15. In fact, arguments from our proof of Theorem 2.15 easily imply that if G is a counterexample to Conjecture 2.26, then G is k-regular and has $g(G) \in \{4, 5\}$. However, we do not see a way to handle these remaining cases without resorting to extensive case analysis (which we have not done).

Significant work has also been done proving lower bounds on $\chi_l(G)$. Brown [8] constructed a graph G with maximum degree k and $\chi_l(G^2) \geq k^2 - k + 1$ whenever $k - 1$ is a prime power. By combining results of Brown [8] and Huxley [28], Miller and Širáň [37] showed that for every $\varepsilon > 0$ there exists a constant c_ε such that for every k there exists a graph G with maximum degree k such that $\chi_l(G^2) \geq k^2 - c_\varepsilon k^{19/12 + \varepsilon}$.

Finally, we can consider the restriction of Theorem 2.15 to planar graphs. If G is a planar subcubic graph, then we know that $\chi_l(G^2) \leq 8$. However, we don't know of any planar graphs for which this is tight. This returns us to the question that motivated much of this research and that remains open.

Question 2.28. *Is it true that every planar subcubic graph G satisfies $\chi_l(G^2) \leq 7$?*

Chapter 3

Discharging

Proofs of coloring results for planar graphs often proceed inductively; they show that each planar graph contains some subgraph H, such that given a coloring of $G \setminus V(H)$, we can extend it to a coloring of G. A simple example of this is the proof that every planar graph is 6-choosable. Since the average degree of every planar graph G is less than 6, G must contain a vertex v of degree at most 5. By the induction hypothesis, $G \setminus \{v\}$ is 6-choosable. Since vertex v has at most 5 neighbors, we can extend the coloring to G.

Rather than a single subgraph H, we often show that G must contain at least one subgraph from some set $\mathcal{H}$. We call a subgraph H a *reducible configuration* if we can show that a coloring of $G \setminus V(H)$ can be extended to G. Usually, we split the proof of a coloring result into two phases: in the first phase we show that every graph must contain some subgraph $H \in \mathcal{H}$, in the second phase we show that each subgraph $H \in \mathcal{H}$ is a reducible configuration. In the first phase, we make no mention of coloring, but instead prove a structural lemma. The greatest difficulty when using this method is usually choosing the set of subgraphs $\mathcal{H}$. Determining this set is a process of trial and error; there is no simple formula for success. However, once we determine our set of subgraphs $\mathcal{H}$, there are powerful tools for proving that every graph contains some subgraph $H \in \mathcal{H}$; the most common of these tools is called the *discharging method*.

In 1905, while working torward a proof of the Four Color Theorem, Wernicke proved the following lemma. If a planar triangulation has minimum degree 5, then it either has an edge with endpoints of degrees 5 and 6 or it has an edge with endpoints both of degree 5; we call these *desired edges*.

We assign a charge $\mu(v) = d(v) - 6$ to each vertex v. The sum of these charges $\sum_{v \in V} d(v) - 6$ equals -12. Thus, if we redistribute the charges but do not change their sum, there must exist a vertex with negative charge; this idea is the basis of the discharging method. Our goal is to redistribute the charge so that any vertex with negative charge is "near" one of the desired edges. We redistribute charge by the following rule: each neighbor of a 5-vertex gives a charge of $1/5$ to the 5-vertex. We apply the rule once, at all vertices simultaneously. After this "discharging," we show that any vertex with negative charge is adjacent to an endpoint of a desired edge. After the discharging phase, we denote the charge at a vertex v by $\mu^*(v)$.

Note that during the discharging phase, the charge at a vertex can decrease by at most $d(v)/5$. Thus, the new charge $\mu^*(v)$ is at least $d(v) - 6 - d(v)/5 = 4d(v)/5 - 6$; this charge is only negative for $d(v) \leq 7$, so we consider 5-vertices, 6-vertices, and 7-vertices. If a 5-vertex or 6-vertex u has negative charge $\mu^*(u)$, then u must be adjacent to a 5-vertex v; so uv is the desired edge. If a 7-vertex u has negative charge, then u must

be adjacent to at least 6 vertices of degree 5. Since the graph is a triangulation, two degree 5 neighbors v and w of vertex u must be adjacent to each other; so vw is the desired edge.

The proof of Wernicke's lemma is a very simple example of discharging. Usually the set of reducible configurations is larger and the dicharging rules are more complex. Additionally, we often assign charge to the faces of a plane graph as well as the vertices.

3.1 Planar graphs with no triangles sharing an edge

All our graphs are finite and without loops or multiple edges. Let G be a plane graph. We use $E(G)$, $V(G)$, $F(G)$, $\Delta(G)$, and $\delta(G)$ to denote the edge set, vertex set, face set, maximum degree, and minimum degree of G, respectively. Where it is clear from context, we use Δ, rather than $\Delta(G)$. We use "j-face" and "j-vertex" to mean faces and vertices of degree j. The degree of a face f is the number of edges along the boundary of f, with each cut-edge being counted twice. The degree of a face f and the degree of a vertex v are denoted by $d(f)$ and $d(v)$. We say a face f or vertex v is *large* when $d(f) \geq 5$ or $d(v) \geq 5$. We use *triangle* to mean 3-face. We use *kite* to mean a subgraph of G formed by two 3-faces that share an edge. We use *element* to mean vertex or face.

A *proper edge-coloring* of G is an assignment of a label to each edge so that no two adjacent edges receive the same label. We call these labels *colors*. A *proper k-edge-coloring* is a proper edge-coloring that uses no more than k colors. An *edge assignment L* is a function on $E(G)$ that assigns each edge e a list $L(e)$ of colors available for use on that edge. An *L-edge-coloring* is a proper edge-coloring with the additional constraint that each edge receives a color appearing in its assigned list. We say that a graph G is *k-edge-choosable* if G has a proper L-edge-coloring whenever $|L(e)| \geq k$ for every $e \in E(G)$. The *chromatic index* of G, denoted $\chi'(G)$, is the least integer k such that G is k-edge-colorable. The *list chromatic index* of G, denoted $\chi'_l(G)$, is the least integer k such that G is k-edge-choosable. In particular, note that $\chi'(G) \leq \chi'_l(G)$. Probably the most fundamental and important result about the chromatic index of graphs (without loops or multiple edges) is:

Theorem 3.1. *(Vizing's Theorem; Vizing [44, 45] and Gupta [19])*

$$\chi'(G) \leq \Delta(G) + 1.$$

Vizing conjectured that Theorem 3.1 could be strengthened by proving the same bound for the list chromatic index:

Conjecture 3.2. *(Vizing's Conjecture; see [34])*

$$\chi'_l(G) \leq \Delta(G) + 1.$$

The most famous open problem about list edge-coloring is the List Coloring Conjecture. Bollobás and Harris [4] believed that Vizing's conjecture could be further strengthened to give:

Conjecture 3.3. *(List Coloring Conjecture; Bollobás and Harris [4])*

$$\chi'_l(G) = \chi'(G).$$

In 1995, Galvin proved that the List Coloring Conjecture is true for bipartite graphs [18]. Borodin [6] showed that the List Coloring Conjecture holds for planar graphs with $\Delta \geq 14$. Borodin, Kostochka, and Woodall [5] improved this result to show that the List Coloring Conjecture holds for planar graphs with $\Delta \geq 12$. Apart from these results, the List Coloring Conjecture has proved very difficult. Fortunately, more progress has been made on Vizing's Conjecture.

Vizing's Conjecture is easy to prove when $\Delta \leq 2$. In general, $\chi'_l(G) \leq 2\Delta - 1$ by coloring greedily in an arbitrary order. Harris [23] showed that if $\Delta \geq 3$, then $\chi'_l(G) \leq 2\Delta - 2$. This implies Vizing's Conjecture when $\Delta = 3$. Juvan et al. [32] confirmed the conjecture when $\Delta = 4$. Vizing's conjecture was also established for other special families of graphs, such as complete graphs [22] and planar graphs with $\Delta \geq 9$ [6]. Wang and Lih [47] proved that Vizing's Conjecture holds for a planar graph G when $\Delta \geq 6$ and G has no two triangles sharing a vertex. Zhang and Wu [55] proved that Vizing's Conjecture holds for a planar graph G when $\Delta \geq 6$ and G has no 4-cycles. Results when $\Delta = 5$ are weaker, since the structural hypotheses are more restrictive. Zhang and Wu [55] showed that a planar graph G is 6-edge-choosable when $\Delta = 5$ and G has no triangles. Wang and Lih [46] showed that a planar graph G is 6-edge-choosable when $\Delta = 5$ and G has no 5-cycles.

We improve these results in several ways. In Section 2, we prove structural results for use in Section 3, where we prove our two main results. We show that Vizing's Conjecture holds for a planar graph G when G contains no kites and $\Delta \geq 6$. This is a strengthening of the result of Wang and Lih [47] and the result of Zhang and Wu [55]. We also show that the List Coloring Conjecture holds for a planar graph G when G contains no kites and $\Delta \geq 9$. Our method, like that of Wang and Lih [47], Zhang and Wu [55], and Borodin [6] is the discharging method. In Section 4, we prove Vizing's Conjecture for a planar graph G when $\Delta = 5$ and G has no 4-cycles; we also prove Vizing's Conjecture for a planar graph G when $\Delta = 5$ and the distance between any two triangles in G is at least 2.

Proofs of coloring results for planar graphs often proceed inductively by showing the existence of certain subgraphs with small degree-sum for the vertices, called "light" copies of these subgraphs. We prove and use several such structural results. For example, we prove that every planar graph G with $\Delta \geq 7$ that contains no kites has an edge whose endpoints have degree-sum at most $\Delta + 2$. For such a graph, G, it follows easily that Vizing's Conjecture holds.

3.1.1 Structural lemmas

The proof of Wernicke's lemma was a simple discharging argument; our next example is more complex: it assigns charge to both vertices and faces, and requires a longer case analysis. In that proof, we used Euler's formula to conclude that our graph G must contain one of the desired subgraphs. In this instance, we assume that our graph G does not contain any of the desired subgraphs; this leads to a contradiction.

Let G be a plane graph. Rewrite Euler's Formula $|F(G)| - |E(G)| + |V(G)| = 2$ as $2|E(G)| - 4|V(G)| + 2|E(G)| - 4|F(G)| = -8$, and then as:

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8.$$

We want to prove that each planar graph G with no kites and maximum degree at least 7 contains a particular type of subgraph. By assuming these subgraphs do not appear, we reach a contradiction in the following

manner. We assign to each element $x \in S$ an initial *charge* $\mu(x)$ defined by $\mu(x) = d(x) - 4$. We will redistribute these charges in a way that preserves the sum of all the charges, and yet makes the new charge $\mu^*(x)$ nonnegative at every element. This produces the obvious contradiction

$$0 \leq \sum_{x \in S} \mu^*(x) \quad = \quad \sum_{x \in S} \mu(x) \quad = \quad \sum_{x \in S}(d(x) - 4) \quad = -8.$$

Our rules for redistributing charges are designed to take advantage of the absence of the forbidden subgraph(s). In the following theorem we forbid an edge uv with $d(u) + d(v) \leq \Delta + 2$, and we also forbid kites. Since each edge uv satisfies $d(u) + d(v) \geq \Delta + 3$, it follows that each neighbor of a 3-vertex is a Δ-vertex. Similarly, since G contains no kites, a vertex v is incident to at most $d(v)/2$ triangles.

Theorem 3.4. *If graph G is planar, G contains no kites, and G has $\Delta \geq 7$, then G has an edge uv with $d(u) + d(v) \leq \Delta + 2$.*

Proof: Assume G is a counterexample. For every edge uv, G must have $d(u) + d(v) \geq \Delta + 3 \geq 10$. Thus, $\delta(G) \geq 3$. We use a discharging argument. We assign to each element x an initial charge $\mu(x) = d(x) - 4$. We use the following two discharging rules, applied simultaneously at all vertices and faces in a single discharging phase:

(R1) Each large vertex v gives a charge of $1/2$ to each incident triangle.

(R2) Each Δ-vertex v gives a charge of $1/3$ to each adjacent 3-vertex.

Now we show that for every element the new charge μ^* is nonnegative.
Consider an arbitrary face f.

- If $d(f) = 3$, then since $d(u) + d(v) \geq 10$ for every edge uv, at least two of the vertices incident to f are large. Thus $\mu^*(f) \geq -1 + 2(1/2) = 0$.

- If $d(f) \geq 4$, then $\mu^*(f) = \mu(f) \geq 0$.

Consider an arbitrary vertex v.

- If $d(v) = 3$, then $\mu^*(v) = -1 + 3(1/3) = 0$, since each neighbor of v is a Δ-vertex.

- If $d(v) = 4$, then $\mu^*(v) = \mu(v) = 0$.

- If $d(v) = 5$, then v is incident to at most 2 triangles, so $\mu^*(v) \geq 1 - 2(1/2) = 0$.

- If $6 \leq d(v) \leq \Delta - 1$, then v is incident to at most $d(v)/2$ triangles. Thus $\mu^*(v) \geq d(v) - 4 - \frac{d(v)}{2}\frac{1}{2} = \frac{3d(v)}{4} - 4 > 0$.

- If $d(v) = \Delta$, then let t be the number of triangles incident to v. For each triangle incident to v, at most one of the vertices of that triangle has degree 3. Thus, if v is incident to t triangles, then $\mu^*(v) \geq d(v) - 4 - t(\frac{1}{2}) - (d(v) - t)(\frac{1}{3}) = \frac{2d(v)}{3} - 4 - \frac{t}{6}$. Since $t \leq \frac{d(v)}{2}$, we get $\mu^*(v) \geq \frac{7d(v)}{12} - 4$. This expression is positive when $d(v) \geq 7$.

$\square$

We will use Theorem 3.4 to show that any planar graph G with $\Delta \geq 7$ that contains no kites is $(\Delta + 1)$-edge-choosable. We would also like to prove an analogous result for the case $\Delta = 6$. To prove such a result, we need the following structural lemma. We say that a triangle is of *type* (a, b, c) if its vertices have degrees a, b, and c. Recall that a face f is *large* if $d(f) \geq 5$.

Lemma 3.5. *If graph G is planar, G contains no kites, and $\Delta = 6$, then at least one of the three following conditions holds:*

(i) *G has an edge uv with $d(u) + d(v) \leq 8$.*

(ii) *G has a 4-face $uvwx$ with $d(u) = d(w) = 3$.*

(iii) *G has a 6-vertex incident to three triangles; two of these triangles are of type $(6, 6, 3)$ and the third is of type $(6, 6, 3)$, $(6, 5, 4)$, or $(6, 6, 4)$.*

Proof: Assume G is a counterexample. For every edge uv, G must have $d(u) + d(v) \geq 9$. Thus, $\delta(G) \geq 3$. We use a discharging argument. We assign to each element x an initial charge $\mu(x) = d(x) - 4$. We use the following three discharging rules:

(R1) Each large face f gives a charge of $1/2$ to each incident 3-vertex.

(R2) Each 5-vertex v gives a charge of $1/2$ to each incident triangle.

(R3) Each 6-vertex v

- gives a charge of $1/3$ to each adjacent 3-vertex that is not incident to any large face.
- gives a charge of $1/6$ to each adjacent 3-vertex that is incident to a large face.
- gives a charge of $1/2$ to each incident triangle that is incident to a 3-vertex or a 4-vertex.
- gives a charge of $1/3$ to each incident triangle that is not incident to a 3-vertex or a 4-vertex.

Now we show that for every element the new charge μ^* is nonnegative.
Consider an arbitrary face f.

- If $d(f) = 3$, then we consider two cases. If f is incident to a 3-vertex or a 4-vertex, then $\mu^*(f) = -1 + 2(1/2) = 0$. If f is not incident to a 3-vertex or a 4-vertex, then $\mu^*(f) \geq -1 + 3(1/3) = 0$.

- If $d(f) = 4$, then $\mu^*(f) = \mu(f) = 0$.

- If $d(f) = 5$, then $\mu^*(f) \geq 1 - 2(1/2) = 0$.

- If $d(f) \geq 6$, then $\mu^*(f) \geq d(f) - 4 - \frac{d(f)}{2}\frac{1}{2} = \frac{3d(f)}{4} - 4 > 0$.

Consider an arbitrary vertex v.

- If $d(v) = 3$, then we consider two cases. If v is incident to a large face, then $\mu^*(v) \geq -1 + 1/2 + 3(1/6) = 0$. If v is not incident to a large face, then $\mu^*(v) = -1 + 3(1/3) = 0$.

- If $d(v) = 4$, then $\mu^*(v) = \mu(v) = 0$.

- If $d(v) = 5$, then $\mu^*(v) \geq 1 - 2(1/2) = 0$.

- If $d(v) = 6$, then we consider separately the four cases where v is incident to zero, one, two, or three triangles. Note that if v is incident to t triangles, then the number of 3-vertices adjacent to v is at most $(6 - t)$.

  ○ If v is incident to no triangles, then $\mu^*(v) \geq 2 - 6(1/3) = 0$.

  ○ If v is incident to one triangle, then we consider two cases. If v is adjacent to at most four 3-vertices, then $\mu^*(v) \geq 2 - (1/2) - 4(1/3) > 0$. If v is adjacent to five 3-vertices, then two of these adjacent 3-vertices lie on a common face, together with v. Since condition (ii) of the present lemma does not hold, this face must be a large face. So $\mu^*(v) \geq 2 - (1/2) - 3(1/3) - 2(1/6) > 0$.

  ○ If v is incident to two triangles, then we consider two cases. If v is adjacent to at most three 3-vertices, then $\mu^*(v) \geq 2 - 2(1/2) - 3(1/3) = 0$. If v is adjacent to four 3-vertices, then two of these adjacent 3-vertices lie on a common face, together with v. Since condition (ii) of the present lemma does not hold, this face must be a large face. So $\mu^*(v) \geq 2 - 2(1/2) - 2(1/3) - 2(1/6) = 0$.

  ○ If v is incident to three triangles, then we consider two cases. If at most one of the triangles is type $(6, 6, 3)$, then $\mu^*(v) \geq 2 - 3(1/2) - 1/3 > 0$. Furthermore, if two of the triangles incident to v are type $(6, 6, 3)$ but the third triangle is not incident to any vertex of degree at most 4, then $\mu^*(v) = 2 - 2(1/2) - 2(1/3) - 1(1/3) = 0$. If two of the triangles are of type $(6, 6, 3)$ and the third triangle is incident to a vertex of degree at most 4, then condition (iii) of the lemma holds.

$\square$

We will apply Theorem 3.4 and Lemma 3.5 to get our first result about edge-choosability. To prove the $(\Delta + 1)$-edge-choosability of a planar graph G that has $\Delta \geq 6$ and that contains no kites, we remove one or more edges of G, inductively color the resulting subgraph, then extend the coloring to G. Intuitively, Theorem 3.4 and Lemma 3.5 do the "hard work." However, it is still convenient to prove the following lemma, which we will apply to the subgraphs of G that arise from this process.

Lemma 3.6. *Let G be a planar graph that contains no kites. If $\Delta \leq 5$, then G has an edge uv with $d(u) + d(v) \leq 8$. If $\Delta = 6$, then G has an edge uv with $d(u) + d(v) \leq 9$.*

Proof: If $\Delta \leq 4$, then each edge uv satisfies $d(u) + d(v) \leq 2\Delta \leq 8$. In that case, the lemma holds trivially. So we must prove the lemma for the cases $\Delta = 5$ and $\Delta = 6$. We handle both cases simultaneously with a discharging argument. Assume G is a counterexample. For every edge uv, G must have $d(u) + d(v) \geq \Delta + 4$. Thus, $\delta(G) \geq 4$. We assign to each element x an initial charge $\mu(x) = d(x) - 4$. We use a single discharging rule:

(R1) Every large vertex v gives a charge of $1/2$ to each incident triangle.

Now we show that for every element the new charge μ^* is nonnegative.

Consider an arbitrary face f.

- If $d(f) = 3$, then f is incident to at least two large vertices, so $\mu^*(f) \geq -1 + 2(1/2) = 0$.

- If $d(f) \geq 4$, then $\mu^*(f) = \mu(f) \geq 0$.

Consider an arbitrary vertex v.

- If $d(v) = 4$, then $\mu^*(v) = \mu(v) = 0$.

- If $d(v) = 5$, then $\mu^*(v) \geq 1 - 2(1/2) = 0$.

- If $d(v) = 6$, then $\mu^*(v) \geq 2 - 3(1/2) > 0$.

$\square$

Theorem 3.7. *If graph G is planar, G contains no kites, and $\Delta \geq 9$, then at least one of the following two conditions holds:*

(i) *G has an edge uv with $d(u) + d(v) \leq \Delta + 1$.*

(ii) *G has an even cycle $v_1 w_1 v_2 w_2 \ldots v_k w_k$ with $d(w_i) = 2$.*

Proof: Assume G is a counterexample. For every edge uv, G must have $d(u) + d(v) \geq \Delta + 2$. Thus, $\delta(G) \geq 2$. Our proof will use a discharging argument, but first we show that if G is a counterexample to Theorem 3.7, then G has more Δ-vertices than 2-vertices.

Let H be the subgraph of G formed by all edges with one endpoint of degree 2 and the other endpoint of degree Δ. Form $\widehat{H}$ from H by contracting one of the two edges incident to each vertex of degree 2 (recall that each neighbor of a 2-vertex in G is a Δ-vertex). Each 2-vertex in G corresponds to an edge in $\widehat{H}$ and each vertex in $\widehat{H}$ corresponds to a Δ-vertex in G. So G has more Δ-vertices than 2-vertices unless $|E(\widehat{H})| \geq |V(\widehat{H})|$.

If $|E(\widehat{H})| \geq |V(\widehat{H})|$, then $\widehat{H}$ contains a cycle. However, a cycle in $\widehat{H}$ corresponds to an even cycle $v_1 w_1 v_2 w_2 \ldots v_k w_k$ in G with $d(v_i) = 2$. Such a cycle in G satisfies condition (ii) and shows that G is not a counterexample to Theorem 3.7. So, G has more Δ-vertices than 2-vertices. We use this fact to design our discharging rules.

We assign to each element x an initial charge $\mu(x) = d(x) - 4$. In addition to the vertices and edges, we create a *bank* that can give and receive charge. The bank has initial charge 0. As with the vertices and edges, we must verify that the final charge of the bank is nonnegative. We use the following three discharging rules:

(R1) Each Δ-vertex and $(\Delta - 1)$-vertex v gives a charge of $1/3$ to each adjacent 2-vertex or 3-vertex.

(R2) Each large vertex v gives a charge of $1/2$ to each incident triangle.

(R3) Each Δ-vertex gives a charge of $4/3$ to the bank.
Each 2-vertex takes a charge of $4/3$ from the bank.

The only rule that effects the bank's charge is (R3). Since G has more Δ-vertices than 2-vertices, the bank's final charge is positve.

Now we show that for every element the new charge μ^* is nonnegative.

Consider an arbitrary face f.

- If $d(f) = 3$, then since $d(u) + d(v) \geq 11$ for every edge uv, at least two of the vertices incident to f are large. Thus $\mu^*(f) \geq -1 + 2(1/2) = 0$.

- If $d(f) \geq 4$, then $\mu^*(f) = \mu(f) \geq 0$.

Consider an arbitrary vertex v.

- If $d(v) = 2$, then $\mu^*(v) = -2 + 2(1/3) + 4/3 = 0$.

- If $d(v) = 3$, then $\mu^*(v) = -1 + 3(1/3) = 0$.

- If $d(v) = 4$, then $\mu^*(v) = \mu(v) = 0$.

- If $d(v) = 5$, then v is incident to at most 2 triangles, so $\mu^*(v) \geq 1 - 2(1/2) = 0$.

- If $6 \leq d(v) \leq \Delta - 2$, then v is incident to at most $\frac{d(v)}{2}$ triangles. Thus $\mu^*(v) \geq d(v) - 4 - \frac{d(v)}{2}(\frac{1}{2}) = \frac{3d(v)}{4} - 4 > 0$.

- If $d(v) = \Delta - 1$, then let t be the number of triangles incident to v. For each triangle incident to v, at most one of the vertices of that triangle has degree 3. Thus, if v is incident to t triangles, then $\mu^*(v) \geq d(v) - 4 - t(\frac{1}{2}) - (d(v) - t)(\frac{1}{3}) = \frac{2d(v)}{3} - 4 - \frac{t}{6}$. Since $t \leq \frac{d(v)}{2}$, we get $\mu^*(v) \geq \frac{7}{12}d(v) - 4$. This expression is positive when $d(v) \geq 8$.

- If $d(v) = \Delta$, then let t be the number of triangles incident to v. For each triangle incident to v, at most one of the vertices of that triangle has degree 3. Thus, if v is incident to t triangles, then $\mu^*(v) \geq d(v) - 4 - \frac{4}{3} - t(\frac{1}{2}) - (d(v) - t)(\frac{1}{3}) = \frac{2d(v)}{3} - \frac{16}{3} - \frac{t}{6}$. Since $t \leq \lfloor \frac{d(v)}{2} \rfloor$, this expression is nonnegative when $d(v) \geq 9$.

$\square$

3.1.2 Application to Edge-Choosability

We now have the necessary tools to prove our two main results.

Theorem 3.8. *Let G be a planar graph that contains no kites. If $\Delta \neq 5$, then $\chi_l'(G) \leq \Delta + 1$. If $\Delta = 5$, then $\chi_l'(G) \leq \Delta + 2$.*

Proof: Let G be a connected graph. Harris [23] and Juvan et al. [32] showed that G is $(\Delta + 1)$-edge-choosable when $\Delta = 3$ and $\Delta = 4$, respectively (even for nonplanar graphs). Thus, we only need to prove the theorem when $\Delta \geq 5$. We consider separately the three cases $\Delta = 5$, $\Delta = 6$, and $\Delta \geq 7$. In each case we proceed by induction on the number of edges. The theorem holds trivially if $|E(G)| \leq 7$. Note that if
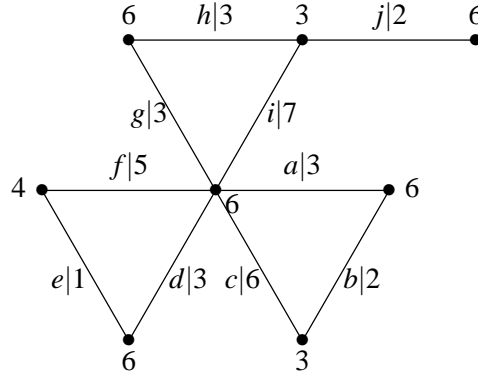
Figure 3.1. The ten remaining uncolored edges. The number at each vertex is the degree of that vertex in G. The number on each edge is the number of colors available to use on that edge after we have chosen colors for all edges not pictured.

$d(u) + d(v) \leq k$, then edge uv is adjacent to at most $k-2$ other edges. We use this fact frequently in the proof.

Consider the case $\Delta(G) = 5$. Let H be a subgraph of G. Since $\Delta(H) \leq 5$, Lemma 3.6 implies that H has an edge uv with $d(u) + d(v) \leq 8$. By hypothesis, $\chi_l'(H - uv) \leq 7$. Since edge uv is adjacent to at most six edges in H, we can extend the coloring to edge uv.

Consider the case $\Delta(G) \geq 7$. Let H be a subgraph of G. Since $\Delta(H) \leq \Delta(G)$, Theorem 3.4 and Lemma 3.6 together imply that H has an edge uv with $d(u) + d(v) \leq \Delta(G) + 2$. By hypothesis, $\chi_l'(H - uv) \leq \Delta(G) + 1$. Since edge uv is adjacent to at most $\Delta(G)$ edges in H, we can extend the coloring to edge uv.

Consider the case $\Delta(G) = 6$. Let H be a subgraph of G. By Lemmas 3.5 and 3.6, we know that one of the three conditions from Lemma 3.5 holds for H. We show that in each case we can remove some set of edges $\widehat{E}$, inductively color the graph $H - \widehat{E}$, then extend the coloring to $\widehat{E}$.

(i) If H has an edge uv with $d(u) + d(v) \leq 8$, then by hypothesis $\chi_l'(H - uv) \leq 7$. Since at most 6 colors are prohibited from use on uv, we can extend the coloring to uv.

(ii) If H has a 4-face $uvwx$ with $d(u) = d(w) = 3$, then let $C = \{uv, vw, wx, xu\}$. By hypothesis $\chi_l'(H - C) \leq 7$. Since each of the four uncolored edges of C has at most 5 colors prohibited, there are at least two colors available to use on each edge of C. Since $\chi_l'(C) = 2$, we can extend the coloring to C. (It is well-known for every even cycle C that $\chi_l'(C) = 2$, but for completeness note that we prove this in case (d) of Lemma 3.10).

(iii) If G has a 6-vertex incident to 3 triangles, two of type $(6, 6, 3)$ and the third of type $(6, 6, 3)$, $(6, 5, 4)$, or $(6, 6, 4)$, then we show how to proceed when the third triangle is type $(6, 6, 4)$; this is the most restrictive case. Let $\widehat{E}$ be the set of edges of all three triangles, plus one additional edge incident to a vertex of degree 3 in one of the triangles. By hypothesis, $\chi_l'(G - \widehat{E}) \leq 7$. We show that we can extend the coloring to $\widehat{E}$.

The ten edges of $\widehat{E}$ are shown in Figure 3.1, along with the number of colors available to use on each edge. We use $L(e)$ to denote the list of colors available for use on edge e after we have chosen colors for all the edges not shown in Figure 3.1. Since $|L(g)| + |L(j)| > |L(h)|$, either there exists some color $\alpha \in L(g) \cap L(j)$ or there exists some color $\alpha \in (L(g) \cup L(j)) \setminus L(h)$. If $\alpha \in L(g) \cap L(j)$, we use color α on edges g and j. Otherwise there exists $\alpha \in (L(g) \cup L(j)) \setminus L(h)$. In this case, use color α on g or j, then use

some other available color on whichever of g and j is uncolored. In either case, we can now color the rest of the edges in the order: e, d, a, b, f, c, i, h.

This completes the proof for the case $\Delta(G) = 6$. $\qquad\square$

Theorem 3.9. *If G is planar, G contains no kites, and $\Delta(G) \geq 9$, then $\chi'_l(G) = \Delta(G)$.*

Proof: Since edges with a common endpoint must receive distinct colors, $\chi'_l(G) \geq \Delta(G)$. So we need to prove that $\chi'_l(G) \leq \Delta(G)$. By induction on the number of edges, we prove that if H is a subgraph of G, then $\chi'_l(H) \leq \Delta(G)$. Our base case is when $\Delta(H) \leq 8$. The result holds for the base case by Theorem 3.8.

Assume that $\Delta(H) \geq 9$. By Theorem 3.7 at least one of the following two conditions holds:

  (i) H has an edge uv with $d(u) + d(v) \leq \Delta(H) + 1$.

  (ii) H has an even cycle $v_1 w_1 v_2 w_2 \ldots v_k w_k$ with $d(v_i) = 2$.

Suppose condition (i) holds. By hypothesis, $\chi'_l(H - uv) \leq \Delta(G)$. Since $d(u) + d(v) \leq \Delta(H) + 1 \leq \Delta(G) + 1$, we have at least one color available to extend the coloring to uv.

Suppose condition (ii) holds. Let C be the even cycle. By hypothesis, $\chi'_l(H - C) \leq \Delta(G)$. After coloring $H - C$, each edge of C has at least two colors available. Since even cycles are 2-choosable, we can extend the coloring to C. $\qquad\square$

3.1.3   Planar graphs with $\Delta(G) = 5$

Proving that a planar graph G with no kites satisfies $\chi'_l(G) \leq \Delta + 1$ seems to be most difficult when $\Delta = 5$. This difficulty is reflected both in the results prior to this paper and in our results. We are unable to show that a planar graph G with no kites and $\Delta = 5$ satisfies $\chi'_l(G) \leq \Delta + 1$. There are two types of weaker conjectures that naturally come to mind. Either we can forbid additional subgraphs (such as a 4-face), or we can require that any two 3-faces of G be further apart. Theorems 3.11 and 3.12 provide results of both types. Before proving these results, in Lemma 3.10 we show that the six configurations in Figure 3.2 are reducible; that is, if $\Delta = 5$ and G contains one of these configurations as a subgraph, then G cannot be a minimal planar graph that is not 6-edge-choosable.

In each of the six cases, we show how to choose colors for the edges of G if one of the reducible configurations is a subgraph of G. Our plan is to choose colors for all edges of $\widehat{G}$, the graph formed by deleting the edges of the reducible configuration, which can be done if G is a minimal counterexample (i.e. no counterexample has fewer edges), then to choose colors for the edges of the reducible configuration. (Usually this final step involves short case analysis.) Our general technique is to show that for some edge e in the reducible configuration, either we can use the same color on two edges that are adjacent to e or we can use a color on some edge adjacent to e that is not in $L(e)$. In the reducible configurations, the number at each vertex is the degree of that vertex in G; the number on each edge is the number of colors available to use on that edge after we have chosen colors for all edges not in the reducible configuration.

Lemma 3.10. *None of the six configurations in Figure 3.2 appear as subgraphs of any minimal planar graph G that has $\Delta = 5$ and is not 6-edge-choosable.*
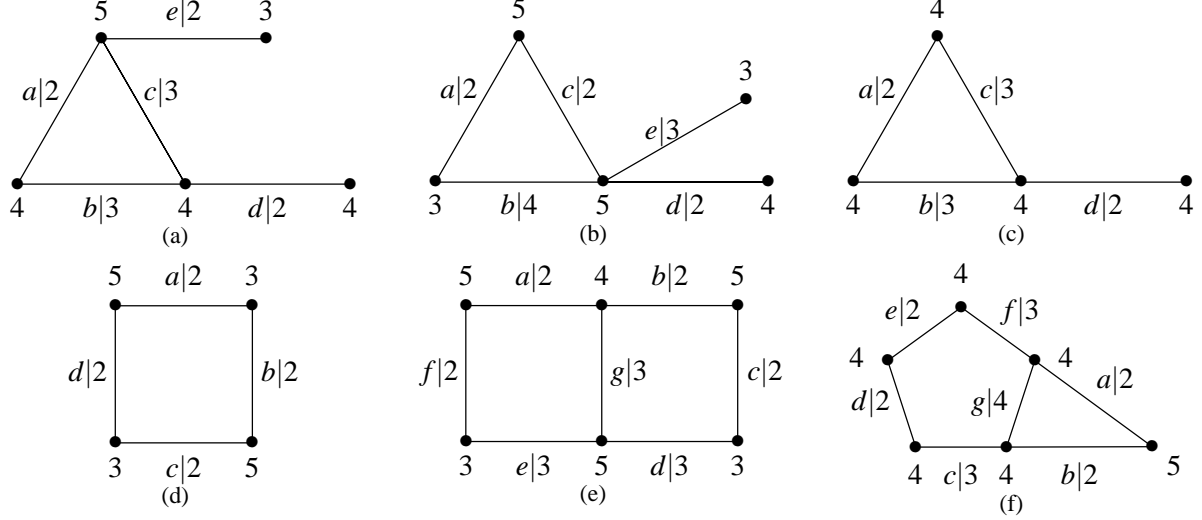
Figure 3.2. Six reducible configurations. The number at each vertex is the degree of that vertex in G. The number on each edge is the number of colors available to use on that edge after we have chosen colors for all edges not in the reducible configuration.

Proof: (a) Since $|L(a)| + |L(d)| > |L(b)|$, either there exists $\alpha \in L(a) \cap L(d)$ or there exists $\alpha \in (L(a) \cup L(d)) \setminus L(b)$. Consider the first case. Use α on edges a and d, then color edges e, c, and b, in that order. Consider the second case. If $\alpha \in L(a) \setminus L(b)$, use α on a, then color edges e, c, d, and b, in that order. Suppose instead that $\alpha \in L(d) \setminus L(b)$. (We assume that $\alpha \notin L(a)$.) If $\alpha \in L(c)$, use α on c, then color e, a, d, and b, in that order. If $\alpha \notin L(c)$, use α on d, then color a, e, c, and b, in that order.

(b) Since $|L(a)| + |L(e)| > |L(b)|$, either there exists $\alpha \in L(a) \cap L(e)$ or there exists $\alpha \in (L(a) \cup L(e)) \setminus L(b)$. Consider the first case. If $\alpha \notin L(d)$, use color α on edges a and e, then color edges c, d, and b, in that order. If $\alpha \in L(d)$, use color α on edges a and d, then color c, e, and b, in that order. Consider the second case. If $\alpha \in L(a) \setminus L(b)$, use α on edge a, then color c, d, e, and b, in that order. Suppose instead that $\alpha \in L(e) \setminus L(b)$. If $\alpha \in L(d)$, then use α on d, then color c, a, e, and b, in that order. If $\alpha \notin L(d)$, then use α on e, then color c, d, a, and b, in that order. Note that if we replace the 4-vertex in (b) with a 3-vertex, no fewer colors are available to use on edge d, so the new configuration is also reducible.

(c) The reducibility of configuration (a) implies the reducibility of configuration (c), since (c) is a subgraph of (a) and each of the edges in (c) has the same number of colors available as the corresponding edge in (a).

(d) If the lists of colors available on all four edges are identical, then we can alternate colors on the cycle (i.e. use color α on edges a and c and use color β on edges b and d). If two lists differ, we may assume (without loss of generality) that there exists $\alpha \in L(a) \setminus L(d)$. Use color α on edge a, then color edges b, c, and d, in that order. In fact, we have proved the stronger statement that every even cycle is 2-choosable.

(e) If $L(a) = L(b)$, use $\alpha \in L(g) \setminus L(a)$ on edge g. The remaining 6-cycle is 2-choosable. So assume $L(a) \neq L(b)$. Choose $\alpha \in L(b) \setminus L(a)$. If $\alpha \notin L(g)$, use α on b, then color edges c and d, in that order. The remaining 4-cycle is 2-choosable. Similarly, if $\alpha \in L(g)$ and $\alpha \in L(d)$, use α on edges b and d, then color c. Again, the remaining 4-cycle is 2-choosable. So assume $\alpha \in L(g)$ and $\alpha \notin L(d)$. Use α on g, then color b, c, a, f, e, and d, in that order.

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(f) Since $|L(a)| + |L(c)| > |L(g)|$, either there exists $\alpha \in L(a) \cap L(c)$, or there exists $\alpha \in (L(a) \cup L(c)) \setminus L(g)$. Consider the first case. Use α on edges a and c, then color edges b, d, e, f, and g, in that order. Consider the second case. If $\alpha \in L(a) \setminus L(g)$, use α on a, then color b, c, d, e, f, and g, in that order. If instead $\alpha \in L(c) \setminus L(g)$, use α on c, then color b, a, d, e, f, and g, in that order. □

In our proofs of Theorem 3.11 and Theorem 3.12, we would like to assume that any possible counterexamples to the theorems do not contain as subgraphs any of the configuration in Figure 3.2. To allow this assumption, in these proofs we argue about a *minimal* counterexample. After proving Theorem 3.11, we learned that it is a special case of result due to Wang and Lih [48]; however, for completeness, we include our proof.

Theorem 3.11. *If G is a planar graph with no 4-cycles and $\Delta(G) = 5$, then G is 6-edge-choosable.*

Proof: Let G be a minimal counterexample to the theorem. If there exists $uv \in E(G)$ with $d(u) + d(v) \leq 7$, we can choose colors for the edges of $G - \{uv\}$ (since G is a minimal counterexample), then choose a color for uv since at most 5 colors are prohibited by adjacent edges. Thus for each edge uv, G must have $d(u) + d(v) \geq 8$. In particular, $\delta(G) \geq 3$. We use a discharging argument. We assign to each vertex or face x the initial charge $\mu(x) = d(x) - 4$. We use the following discharging rules:

(R1) For each large face f,

- transfer a charge of 1/2 from f to each incident 3-vertex.
- transfer a charge of 1/4 from f to each incident 4-vertex that is incident to a triangle adjacent to f.

(R2) For each vertex v of degree 4,

- transfer a charge of 1/4 from v to each incident $(5,4,4)$ triangle.
- transfer a charge of 1/2 from v to each incident $(4,4,4)$ triangle.

(R3) For each vertex v of degree 5, transfer a charge of 1/2 from v to each incident triangle.

Now we show that for every vertex and face μ^* is nonnegative. Throughout the proof we implicitly use the facts that G has no 4-faces and that G does not have two adjacent 3-faces (which imply a 4-cycle). Consider an arbitrary vertex v.

- If $d(v) = 3$, then v is adjacent to at least two large faces, so $\mu^*(v) \geq -1 + 2(1/2) = 0$.

- If $d(v) = 4$, then we consider three cases, depending on the triangles incident to v. Note that v is incident to at most two triangles. Furthermore, if v is incident to at least one triangle f, then v is also incident to two large faces that are adjacent to f; each of these large faces gives v a charge of 1/4.

  ○ If v is incident to no triangles, then $\mu^*(v) = \mu(v) = 0$.

  ○ If v is incident to at least one triangle, but v is not incident to a type $(4,4,4)$, then $\mu^*(v) \geq 0 + 2(1/4) - 2(1/4) = 0$

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○ If v is incident to a type (4,4,4) and also incident to another triangle that receives charge from v (type (4,4,4) or (5,4,4)), then G contains the reducible configuration in Figure 3.2(c). So if v gives charge to a type (4,4,4), then v does not give charge to any other triangle. Thus $\mu^*(v) = 0 + 2(1/4) - 1(1/2) = 0$.

- If $d(v) = 5$, then v is adjacent to at most two triangles (otherwise G contains a 4-cycle). Hence $\mu^*(v) \geq 1 - 2(1/2) = 0$.

Consider an arbitrary face f.

- If $d(f) = 3$, then since $d(u) + d(v) \geq 8$ for each edge uv, there are five types of 3-faces we must consider: (5,5,5), (5,5,4), (5,5,3), (5,4,4), and (4,4,4).

  ○ If f is type (5,5,x) (for some value of x), then $\mu^*(f) \geq -1 + 2(1/2) = 0$.
  ○ If f is type (5,4,4), then $\mu^*(f) = -1 + 1(1/2) + 2(1/4) = 0$.
  ○ If f is type (4,4,4), then $\mu^*(f) = -1 + 3(1/2) > 0$.

- If $d(f) = 4$, then we contradict the present theorem's hypothesis that G contains no 4-cycles.

- If $d(f) = 5$, then we consider three cases. Note that face f is incident to at most two 3-vertices.

  ○ If f is incident to no 3-vertices, then since G does not contain the reducible configuration in Figure 3.2(f), we may assume that f gives charge to at most four 4-vertices. Thus $\mu^*(f) \geq 1 - 4(1/4) = 0$.
  ○ If f is incident to one 3-vertex, then f is incident to at most two 4-vertices. Thus $\mu^*(f) \geq 1 - 1(1/2) - 2(1/4) = 0$.
  ○ If f is incident to two 3-vertices, then f is incident to no 4-vertices. Thus $\mu^*(f) = 1 - 2(1/2) = 0$.

- If $d(f) \geq 6$, then let t be the number of 3-vertices incident to f. If a 3-vertex is incident to f, the clockwise neighbor of that 3-vertex along face f must be a 5-vertex. Hence, if f is incident to t 3-vertices, then the maximum number of 4-vertices incident to f is $d(f) - 2t$. So $\mu^*(f) \geq d(f) - 4 - t(1/2) - (d(f) - 2t)(1/4) = 3d(f)/4 - 4 > 0$.

$\square$

Before we prove our final result, we introduce one more definition. We say that vertices u and w are *successive* neighbors of v if w is the next neighbor of v that we encounter when we start at u and proceed in a clockwise (or counterclockwise) manner around v. In particular, each neighbor of a vertex v has two successive neighbors (with respect to v).

Theorem 3.12. *Let G be a planar graph with $\Delta(G) = 5$. If the distance between any two triangles in G is at least 2, then G is 6-edge-choosable.*

Proof: Let G be a minimal counterexample to the theorem. If there exists an edge uv with $d(u) + d(v) \leq 7$, we can choose colors for the edges of $G - \{uv\}$ (since G is a minimal counterexample), then choose a color for uv since at most 5 colors are prohibited by adjacent edges. Thus for each edge uv, G must have $d(u) + d(v) \geq 8$. In particular, $\delta(G) \geq 3$. We use a discharging argument. We assign to each vertex or face x the initial charge $\mu(x) = d(x) - 4$. We use the following discharging rules:

(R1) For each large face f, transfer a charge of $1/2$ from f to each incident 3-vertex.

(R2) For each vertex v of degree 5,

- transfer a charge of $1/3$ from v to each adjacent 3-vertex that is not incident to any large face.
- transfer a charge of $1/6$ from v to each adjacent 3-vertex that is incident to a large face.

(R3) For each vertex v of degree 5 that is not incident to any triangle,

- transfer a charge of $1/6$ from v to each adjacent 4-vertex that is incident to a triangle.
- transfer a charge of $1/6$ from v to each adjacent 5-vertex w that is incident to a triangle unless both successive neighbors of w (with respect to v) are 3-vertices.

(R4) For every vertex v of degree 4 or 5, after all other applicable rules have been applied, transfer any positive charge remaining at v to its incident triangle (if v is incident to a triangle).

Now we show that for every vertex and face μ^* is nonnegative. We frequently make use of the following fact. If vertex v is incident to triangle T, no neighbor of v is incident to any triangle other than T. We refer to the neighbors of v that are not incident to T as *off-triangle* neighbors.

Consider an arbitrary face f.

- If $d(f) = 3$, we do a case analysis based on the degrees of the vertices incident to f.

  - If f is a type (4,4,4), let v be a 4-vertex on f. Each off-triangle neighbor of v must be a 5-vertex (since G does not contain the reducible configuration in Figure 3.2(c)). Thus, v receives a charge of $1/6$ from each of its off-triangle neighbors and $\mu^*(f) = -1 + 6(1/6) = 0$.

  - If f is incident to a 5-vertex, we consider the case later, when we consider all 5-vertices.

- If $d(f) = 4$, then $\mu^*(f) = \mu(f) = 0$.

- If $d(f) = 5$, then f is incident to at most two 3-vertices. Thus $\mu^*(f) \geq 1 - 2(1/2) = 0$.

- If $d(f) \geq 6$, then $\mu^*(f) \geq d(f) - 4 - (d(f)/2)(1/2) = 3d(f)/4 - 4 > 0$.

Consider an arbitrary vertex v.

- If $d(v) = 3$, then we consider two cases.

  - If v is not incident to any large face, then $\mu^*(v) = -1 + 3(1/3) = 0$.
  - If v is incident to a large face, then $\mu^*(v) \geq -1 + (1/2) + 3(1/6) = 0$.

- If $d(v) = 4$, then $\mu^*(v) = \mu(v) = 0$.

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- If $d(v) = 5$, then we do a case analyis depending on the type of triangle incident to v, with a separate case if no triangle is incident to v. At the same time that we show that $\mu^*(v) \geq 0$ we will also show that $\mu^*(f) \geq 0$ for the triangle f incident to v.

  ○ If v is not on any triangle, then we consider four cases depending on how many 3-vertices are adjacent to v.

    ∗ If v is adjacent to at most one 3-vertex, then $\mu^*(v) \geq 1 - 4(1/6) - 1(1/3) = 0$.

    ∗ If v is adjacent to two 3-vertices, we consider two cases. If the two 3-vertices are succesive, then they both lie on a large face (since G does not contain the reducible configuration in Figure 3.2(d)), so $\mu^*(v) \geq 1 - 5(1/6) > 0$. If the two 3-vertices are not successive, then let u be the neighbor of v between the 3-vertices. If u is a 5-vertex, then $\mu^*(v) \geq 1 - 2(1/3) - 2(1/6) = 0$. If u is a 4-vertex, then one of the 3-vertices adjacent to v must be incident to a large face (since G does not contain the reducible configuration in Figure 3.2(e)). Thus $\mu^*(v) \geq 1 - 4(1/6) - 1(1/3) = 0$.

    ∗ If v is adjacent to three 3-vertices, then two 3-vertices must be successive. Since G does not contain the reducible configuration in Figure 3.2(d), these 3-vertices must lie on a large face. So $\mu^*(v) \geq 1 - 4(1/6) - 1(1/3) = 0$.

    ∗ If v is adjacent to at least four 3-vertices, then each 3-vertex is incident to a large face, so $\mu^*(v) \geq 1 - 5(1/6) > 0$.

  ○ If v is incident to a triangle, we consider four cases depending on whether v is incident to a type $(5,5,5)$, $(5,5,4)$, $(5,5,3)$, or $(5,4,4)$.

In each of the cases below, let v be a 5-vertex, incident to a triangle f. We show that in each case $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$. Our calculations of $\mu^*(v)$ are before v transfers any charge to f (but after all other applicable rules) and thus represent the charge that v transfers to f.

Case (5,5,5): If f is a type $(5,5,5)$, we show that v transfers a charge of at least $1/3$ to f; and thus $\mu^*(f) \geq -1 + 3(1/3) = 0$. If v is adjacent to at most two 3-vertices, then $\mu^*(v) \geq 1 - 2(1/3) = 1/3$. If v is adjacent to three 3-vertices, then each adjacent 3-vertex has a 3-vertex as a successive neighbor. Since G does not contain the reducible configuration in Figure 3.2(d), each 3-vertex adjacent to v is incident to a large face (and thus receives only a charge of $1/6$ from v). So $\mu^*(v) \geq 1 - 3(1/6) > 1/3$.

Case (5,5,4): If f is a type $(5,5,4)$, then let w be the 4-vertex incident to f and let x be the off-triangle neighbor of w that is incident to a face (call it $\hat{f}$) that is incident to v. We show that the charge received by f from v and x totals at least $1/2$. Since f is incident to two 5-vertices, $\mu^*(f) \geq -1 + 2(1/2) = 0$. We consider three cases. If v is adjacent to at most one 3-vertex, then $\mu^*(v) \geq 1 - 1(1/3) > 1/2$. If v is adjacent to three 3-vertices, then each adjacent 3-vertex must be incident to a large face, so $\mu^*(v) \geq 1 - 3(1/6) = 1/2$. If v is adjacent to two 3-vertices, then we consider two sub-cases. If either adjacent 3-vertex is incident to a large face, then $\mu^*(v) \geq 1 - 1(1/3) - 1(1/6) = 1/2$. If each adjacent 3-vertex is not incident to a large face, then let y be the 3-vertex that is adjacent to v and that is incident to $\hat{f}$. Since $\hat{f}$ is not a large face or a triangle, $\hat{f}$ must be a 4-face. Since y is a 3-vertex (and is adjacent to x), x must be a 5-vertex. Hence, w receives a charge of $1/6$ from x. Since $\mu^*(v) \geq 1 - 2(1/3) = 1/3$, the total charge f receives from v and x (via w) is at least $1/3 + 1/6 = 1/2$.

Case (5,5,3): If f is a type $(5,5,3)$, then we show that v transfers a charge of at least $1/2$ to f and thus $\mu^*(f) \geq -1 + 2(1/2) = 0$. Let w be the 3-vertex on the triangle; we consider two cases. If v is not adjacent to any 3-vertices besides w, then $\mu^*(v) \geq 1 - 1(1/3) > 1/2$. If v is adjacent to a 3-vertex besides w, then v is adjacent to exactly one 3-vertex, and v is not adjacent to any 4-vertices (since G does not contain the reducible configuration in Figure 3.2(b)); we consider two sub-cases. If w is incident to a large face, then $\mu^*(v) \geq 1 - 1(1/3) - 1(1/6) = 1/2$. If w is not incident to a large face, then let $\hat{f}$ be the 4-face that is incident to both v and w. Let x be the other neighbor of w on $\hat{f}$ and let y be the other neighbor of v on $\hat{f}$. Both x and y are 5-vertices. To see this, note that y cannot be a 3-vertex, since G does not contain the reducible configuration in Figure 3.2(d) and y cannot be a 4-vertex since v is not adjacent to any 4-vertices. So by (R3), y gives a charge of $1/6$ to v. Thus, $\mu^*(v) \geq 1 + 1/6 - 2(1/3) = 1/2$.

Case (5,4,4): If f is a type $(5,4,4)$, then we consider two cases. If v is adjacent to no 3-vertices, then v gives a charge of 1 to f, so $\mu^*(f) \geq -1 + 1 = 0$ and $\mu^*(v) = 1 - 1 = 0$. If v is adjacent to at least one 3-vertex, then we show that v always gives a charge of at least $1/3$ to f; we consider two sub-cases. If v is adjacent to at most two 3-vertices, then $\mu^*(v) \geq 1 - 2(1/3) = 1/3$. If v is adjacent to three 3-vertices, then each 3-vertex must be incident to a large face, so $\mu^*(v) \geq 1 - 3(1/6) > 1/3$. Since G does not contain the reducible configuration in Figure 3.2(a), all off-triangle neighbors of the two 4-vertices incident to f must be 5-vertices. Each of these four 5-vertices gives a charge of $1/6$ to one of the 4-vertices, so $\mu^*(f) \geq -1 + 4(1/6) + 1(1/3) = 0$. $\qquad\square$

3.2 Planar subcubic graphs with large girth

In this section we use discharging to prove upper bounds on the list-chromatic numbers of squares of planar graphs with large girth. More precisely, given k, we seek the smallest threshold on the girth of G that will guarantee that G^2 is k-choosable.

Define a graph G to be k-minimal if G^2 is not k-choosable, but the square of every proper subgraph of G is k-choosable. A *configuration* is a graph that may arise as an induced subgraph of G. Let a configuration be *k-reducible* if it cannot appear in a k-minimal graph (we will be interested in the cases $k = 6$ and $k = 7$).

As a further refinement, we say that a configuration is $6'$-*reducible* if it cannot appear in a 6-minimal graph with girth at least 7. Note that when $k \geq 4$, a k-minimal subcubic graph contains no 1-vertex (if $d_G(x) = 1$ and G is k-minimal, then $d_{G^2}(x) = d_G(y) \leq 3$, where y is the neighbor of x in G; since $G^2 - x = (G - x)^2$ when $d_G(x) = 1$, we can choose colors for $(G \setminus \{x\})^2$ from its lists and have a color remaining available in $L(x)$ to complete a proper coloring of G^2). Therefore, we assume henceforth that $\delta(G) \geq 2$.

The definition of k-minimal requires that $(G - S)^S$ is k-choosable whenever $S \subseteq V(G)$, but it does not require the stronger statement that $G^2 - S$ is k-choosable ($G^2 - S$ may have edges within $N_G(S)$ that do not appear in $(G - S)^2$. This is a subtle but important distinction. To avoid trouble, we will consider only reducible configurations H such that $G^2 \setminus V(H) = (G \setminus V(H))^2$. Otherwise, we may face difficulties as in the next paragraph.

Here we give a fallacious proof that $\chi_l(G^2) \leq 7$ for every subcubic planar graph G with girth at least 6. A vertex of degree at most 2 forms a 7-reducible configuration in G, since it has degree at most 6 in G^2. Let G be a 7-minimal subcubic planar graph with girth at least 6. Every planar graph with girth at least 6 has a vertex v of degree at most 2 (by Lemma 1.2). If we can choose color for $G^2 \setminus \{v\}$, then we can extend the coloring to v. Unfortunately, k-minimality only implies that we can choose colors for $(G - v)^2$, not for

$G^2 - v$, which may have one additional edge joining the vertices u and w of $N_G(v)$. Hence we cannot apply the induction hypothesis.

In Section 3.2.1 and Section 3.2.2 we obtain girth thresholds for 7-choosability and 6-choosability of the square of a subcubic planar graph. The outlines of the two proofs are very similar. In each, we obtain four reducible configurations (forbidden in k-minimal graphs with appropriate girth). With $g = 7$ when $k = 7$ and $g = 9$ when $k = 6$, we show that $\mathrm{mad}(G) \geq \frac{2g}{g-2}$ for a k-minimal graph that avoids the reducible configurations. On the other hand, the well-known lemma we proved as Lemma 1.2 states that $\mathrm{mad}(G) < \frac{2g}{g-2}$. The contradiction prohibits k-minimal graphs and proves the theorem.

3.2.1 Planar subcubic graphs with girth at least 7

We now prove that $\chi_l(G^2) \leq 7$ when G is a subcubic planar graph with girth at least 7. As observed above, it suffices to obtain four 7-reducible configurations such that every subcubic graph G with $\mathrm{mad}(G) < \frac{14}{5}$ contains at least one of them.

Lemma 3.13. *The following configurations are 7-reducible (they cannot appear in a 7-minimal subcubic graph).*
Configuration 1*: two adjacent 2-vertices.*
Configuration 2*: two 2-vertices with a common neighbor of degree 3.*
Configuration 3*: two adjacent 3-vertices having distinct 2-vertices as neighbors.*
Configuration 4*: a 3-vertex whose neighbors all have degree 3 and have distinct 2-valent neighbors.*

Proof: Configuration 1: Let v_1 and v_2 be two adjacent 2-vertices, and let $H = G - v_1 - v_2$. By the minimality of G, H^2 has a proper coloring from any lists of size 7. We have colored at most five vertices of $N_{G^2}(v_i)$, for each i. Hence we can choose colors from $L(v_1)$ and $L(v_2)$ in turn to extend the coloring to G^2.

Configuration 2: Let v_1 and v_2 be 2-vertices with a common neighbor u of degree 3, and let $H = G - \{v_1, v_2, u\}$. Again H^2 has a proper coloring from its lists. We have colored at most four vertices in $N_{G^2}(v_i)$ and at most five vertices in $N_{G^2}(u)$. Choosing colors for the remaining vertices in the order u, v_1, v_2 allows us to extend the coloring to G^2.

Configuration 3: Let u_1 and u_2 be adjacent 3-vertices having distinct neighbors v_1 and v_2 of degree 2, respectively, and let $H = G - \{v_1, v_2, u_1, u_2\}$. Again H^2 has a proper coloring from its lists. For each of the four remaining vertices, we have colored at most four of the vertices in its neighborhood in N_{G^2}. If we complete the coloring in the order u_1, u_2, v_1, v_2, then when we reach each vertex, we have colored at most six vertices in its neighborhood in G^2, and a color remains available in its list.

Configuration 4: Let w be a 3-vertex have neighbors u_1, u_2, and u_3 of degree 3, adjacent to distinct vertices v_1, v_2, and v_3 of degree 2, respectively. Let $H = G - \{v_1, v_2, v_3, u_1, u_2, u_3, w\}$. Again H^2 has a proper coloring from its lists. For w, u_i, or v_i, we have colored at most three vertices, four vertices, or four vertices from its neighborhood in G^2, respectively. We choose colors for each u_i and then w and then each v_i. When we reach each of these vertices, we have colored at most six vertices in its neighborhood in G^2, and a color remains available in its list. □

Lemma 3.14. *Let G be a minimal graph such that $\chi_l(G^2) > 7$. For $v \in V(G)$, let $M_1(v)$ and $M_2(v)$ be the number of 2-vertices at distance 1 and distance 2 from v in G, respectively. If v is a 3-vertex, then $2M_1(v) + M_2(v) \leq 2$. If v is a 2-vertex, then $2M_1(v) + M_2(v) = 0$.*

Proof: A 7-minimal graph cannot contain one of the four 7-reducible configurations obtained in Lemma 3.13. We show that if G has a vertex v such that the quantity $2M_1(v) + M_2(v)$ is larger than claimed, then G contains such a configuration.

If v is a 2-vertex and $M_1(v) + M_2(v) > 0$, then G contains Configuration 1 or Configuration 2. Hence $2M_1(v) + M_2(v) = 0$ for every 2-vertex v. If v is a 3-vertex, then $M_1(v) > 1$ yields Configuration 2. If $M_1(v) = 1$ and $M_2(v) \geq 1$, then G contains Configuration 3. If $M_1(v) = 0$ and $M_2(v) \geq 3$, then G contains Configuration 4. Hence $2M_1(v) + M_2(v) \leq 2$. $\qquad\square$

Theorem 3.15. *If G is a subcubic graph with $Mad(G) < \frac{14}{5}$, then $\chi_l(G^2) \leq 7$.*

Proof: Let G be a minimal counterexample to the theorem. By Lemma 3.14, each 3-vertex v satisfies $2M_1(v) + M_2(v) \leq 2$ and each 2-vertex v satisfies $2M_1(v) + M_2(v) = 0$. We show that these bounds require $\mathrm{mad}(G) \geq \frac{14}{5}$. We use discharging to average out the vertex degrees, raising the degree "assigned" to 2-vertex until every vertex is assigned at least $14/5$. The initial charge $\mu(v)$ for each vertex v is its degree. We use a single discharging rule:

> **R1**: Each 3-vertex gives $\frac{1}{5}$ to each 2-vertex at distance 1 and gives $\frac{1}{10}$ to each 2-vertex at distance 2.

Let $\mu^*(v)$ be the resulting charge at v. Each 2-vertex has distance at least 3 from every other 2-vertex. If $d(v) = 2$, we therefore have $\mu^*(v) = 2 + 2(\frac{1}{5}) + 4(\frac{1}{10}) = \frac{14}{5}$. Since $2M_1(v) + M_2(v) \leq 2$ when $d(v) = 3$, we obtain $\mu^*(v) = 3 - \frac{1}{5}M_1(v) - \frac{1}{10}M_2(v) = 3 - \frac{1}{10}(2M_1(v) + M_2(v)) \geq 3 - \frac{1}{5} = \frac{14}{5}$ in this case. Since each vertex now has charge at least $\frac{14}{5}$, the average degree is at least $\frac{14}{5}$, a contradiction. $\qquad\square$

Corollary 3.16. *If G is a planar subcubic graph with girth at least 7, then $\chi_l(G^2) \leq 7$.*

Proof: Lemma 1.2 yields $\mathrm{mad}(G) < \frac{14}{5}$. By Theorem 3.15, this implies that $\chi_l(G^2) \leq 7$. $\qquad\square$

3.2.2 Planar subcubic graphs with girth at least 9

We now prove that $\chi_l(G^2) \leq 6$ when G is a subcubic planar graph with girth at least 9. Recall that a configuration is $6'$-reducible if it cannot appear in a 6-minimal graph with girth at least 7. As observed above, it suffices to obtain a set of $6'$-reducible configurations such that every subcubic graph G with $\mathrm{mad}(G) < \frac{14}{5}$ contains at least one of them.

Note that adjacent vertices of degree 2 form a reducible configuration, since deleting them from a $6'$-minimal graph leaves a graph H such that H^2 is 6-choosable, and for each of the deleted vertices only four neighbors in G^2 are colored when colors are chosen for H^2 from its lists. Hence we may assume that G has no adjacent 2-vertices.

We will prove that also the four configurations shown in Figures 3.3a, 3.3b, 3.4a, and 3.4b are $6'$-reducible. We begin with a definition: If v is a 3-vertex, then we say that v is of *class i* if v has i neighbors of degree 2.
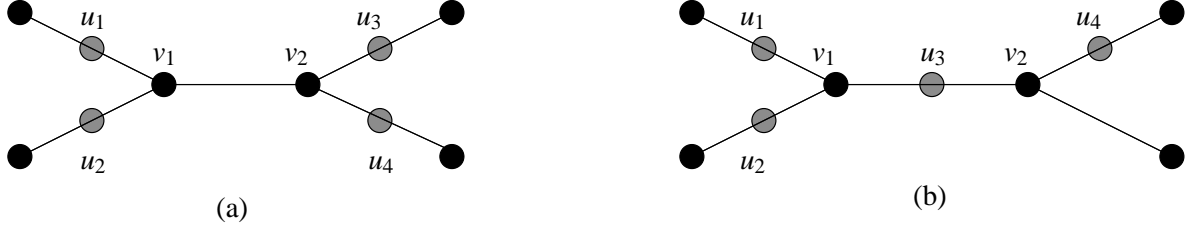
Figure 3.3. Two $6'$-reducible subgraphs. (a) Two adjacent class 2 vertices v_1 and v_2. (b) A class 3 vertex v_1 and a class 2 vertex v_2 at distance 2.

Lemma 3.17. *Adjacent class 2 vertices, with their incident 2-neighbors, form a $6'$-reducible configuration, shown on the left in Figure 3.3.*

Proof: Let v_1 and v_2 be adjacent class 2 vertices. Let u_1 and u_2 be the other neighbors of v_1, and let u_3 and u_4 be the other neighbors of v_2. Let $H = G - \{v_1, v_2, u_1, u_2, u_3, u_4\}$. By the minimality of G, H^2 has a proper coloring from any lists of size 6. We have colored three vertices of $N_{G^2}(u_i)$ and two vertices of $N_{G^2}(v_j)$ for each i and j. Since G has girth at least 7, each of u_1 and u_2 has distance 3 from each of u_3 and u_4. For each remaining vertex x, let $L'(x)$ be the list of remaining available colors at x.

We have $|L'(u_i)| \geq 3$ and $|L'(v_j)| \geq 4$ for each i and j; by discarding colors if necessary, we may assume equality. These sizes are not quite big enough to color greedily in a specified order. However, we can choose a color for v_1 that leaves three colors available at u_1. After assigning this color to v_2, we have three available colors remaining at each of u_1 and v_2, but only two at each of $\{u_2, u_3, u_4\}$. By choosing colors at vertices in the order u_3, u_4, v_2, u_2, u_1, we complete the extension to an L-coloring of G^2. $\qquad\square$

Lemma 3.18. *A configuration that consists of two 3-vertices with a common neighbor u of degree 2, plus all their incident 2-vertices, is $6'$-reducible if one of the 3-vertices is class 3 and the other is class 2 or class 3. (This configuration is shown on the right in Figure 3.3.) Furthermore, if G is any graph containing this configuration and L is a 6-uniform list assignment such that $G^2 - u$ has an L-coloring, then G^2 has L-colorings in which u has distinct colors.*

Proof: Let v_1 and v_2 be such 3-vertices, with u_3 being their common neighbor. Let u_1 and u_2 be the other neighbors of v_1 (having degree 2). Let u_4 be another 2-vertex adjacent to v_2. Let $H = G - \{v_1, v_2, u_1, u_2, u_3, u_4\}$. By the minimality of G, H^2 has an L-coloring. Let $L'(x)$ denote the list of remaining available colors for each x in $V(G) - V(H)$. Note that $|L'(u_1)| \geq 3$, $|L'(u_2)| \geq 3$, $|L'(u_3)| \geq 5$, $|L'(u_4)| \geq 2$, $|L'(v_1)| \geq 4$, and $|L'(v_2)| \geq 2$. We may assume that equality holds for each. (Since G has girth at least 7, note that u_4 has distance at least 3 from each of u_1, u_2, and v_1.)

Since $|L'(v_1)| = 4$ and $|L'(u_1)| = 3$, we can choose for v_1 a color c in $L'(v_1) - L'(u_1)$. Next choose a color for v_2 and then for u_4. At this point, $\{u_3, u_2, u_1\}$ remain to be colored, with remaining lists of sizes $2, 2, 3$, respectively. We can use either remaining color on u_3 and then choose colors for u_2 and u_1. We have produced L-colorings having distinct colors at u_3. $\qquad\square$

We use the term *H-configuration* to denote the configuration consisting of a class 1 vertex adjacent to two class 2 vertices, plus all the 2-vertices adjacent to these three. An *H*-configuration is shown on the left in Figure 3.4).
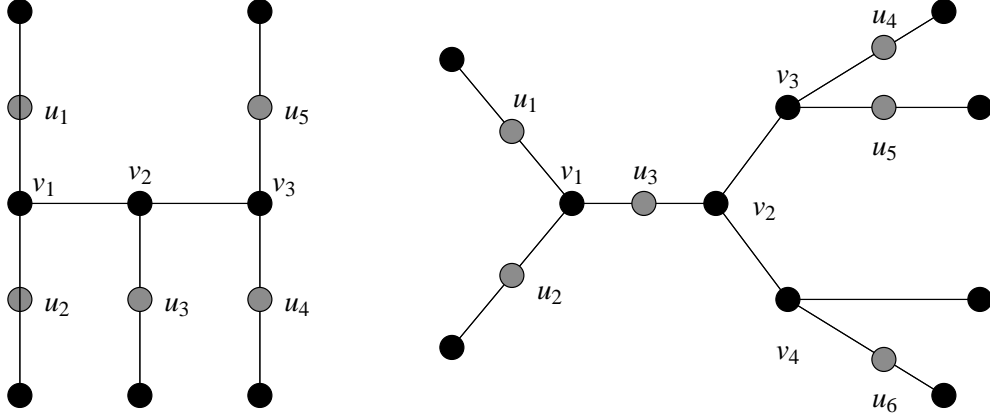
Figure 3.4. An H-configuration and a Y-configuration; both configurations are $6'$-reducible. (a) An H-configuration: a class 1 vertex v_2 is adjacent to two class 2 vertices v_1 and v_3. (b) A Y-configuration: a class 1 vertex v_2 is adjacent to a class 2 vertex v_3 and a class 1 vertex v_4, and is distance two from a class 3 vertex v_1.

Lemma 3.19. *An H-configuration is $6'$-reducible.*

Proof: Let G be a $6'$-minimal graph, and let L be a 6-uniform list assignment for G. Let $v_1, v_2, v_3, u_1, u_2,$ u_3, u_4, u_5 be the vertices of an H-configuration in G as labeled in Figure 3.4). Let H be the subgraph of G obtained by deleting the vertices of the configuration. By the minimality of G, H^2 has an L-coloring. Let $L'(x)$ denote the list of remaining available colors for each uncolored vertex x in G. Note that $|L'(u_i)| \geq 3$, $|L'(v_1)| \geq 4$, $|L'(v_3)| \geq 4$, and $|L'(v_2)| \geq 5$. We may assume that equality holds. Since $|L'(v_2)| > |L'(u_3)|$, we can choose for v_2 a color c in $L'(v_2) - L'(u_3)$. This reduces the lists other than $L'(u_3)$ by 1, but the remaining lists are big enough to choose colors for vertices in the order $u_1, u_2, v_1, u_3, v_3, u_4, u_5$. This extends the L-coloring to G^2. $\qquad\square$

We use the term *Y-configuration* to denote the configuration consisting of four 3-vertices v_1, v_2, v_3, v_4 and their adjacent neighbors of degree 2, where the classes of the 3-vertices are $3, 1, 2, 1$, respectively, with v_3, v_2, v_4 forming a path in order and v_1 having a common neighbor with v_2. The configuration is shown on the right in Figure 3.4.

Lemma 3.20. *A Y-configuration is $6'$-reducible.*

Proof: Let G be a $6'$-minimal graph, and let L be a 6-uniform list assignment for G. Let $v_1, v_2, v_3, v_4, u_1, u_2,$ u_3, u_4, u_5, u_6 be the vertices of an L-configuration in G as labeled in Figure 3.4). Let $H = G - \{v_1, u_1, u_2, u_3\}$. By the minimality of G, H^2 has an L-coloring. Note that the four uncolored vertices form a clique in G^2. Let $L'(x)$ denote the list of remaining available colors for each uncolored vertex x in G. Each $L'(x)$ has size at least 3. Furthermore, the coloring extends to an L-coloring of G^2 unless the remaining lists all have size 3 and are equal.

Note that after vertices v_1, u_1, u_2, and u_3 are deleted, the subgraph induced by vertices v_2, v_3, v_4, u_4, u_5, and u_6 is the same subgraph shown to be $6'$-reducible in Lemma 3.18. By Lemma 3.18, H^2 has a recoloring such that v_2 gets a different color than it currently has. Under this recoloring of H^2, the lists of available

colors for v_3, u_1, u_2, and u_3 are no longer identical. Hence, the recoloring of H^2 extends to an L-coloring of G^2. □

Theorem 3.21. *If G is a subcubic graph with $Mad(G) < \frac{18}{7}$ and girth at least 7, then $\chi_l(G^2) \leq 6$.*

Proof: Let G be a $6'$-minimal graph. As in Theorem 3.15, it suffices to show that if G contains none of the $6'$-reducible configurations in Lemmas 3.17, 3.18, 3.19, or 3.20, then $Mad(G) \geq \frac{18}{7}$. Again we use a discharging argument with initial charge equal to degree and adjust the charge so that each vertex retains charge at least $\frac{18}{7}$. We use three discharging rules.

> **R1**: Each 3-vertex gives $\frac{2}{7}$ to each adjacent 2-vertex.

> **R2**: Each class 0 vertex gives $\frac{1}{7}$ to each adjacent 3-vertex.

> **R3**: Each class 1 vertex gives $\frac{1}{7}$ to each adjacent class 2 vertex and gives $\frac{1}{7}$ to each class 3 vertex at distance 2.

Let $\mu * (v)$ denote the resulting charge at vertex v.

We observed that a $6'$-minimal graph has no adjacent 2-vertices. Therefore, $\mu^*(v) = 2 + 2(\frac{2}{7}) = \frac{18}{7}$ when $d(v) = 2$, and hence it suffices to consider 3-vertices.

If v is class 0, then $\mu^*(v) = 3 - 3(\frac{1}{7}) = \frac{18}{7}$.

If v is class 2, then by Lemma 3.17 vertex v is adjacent to a class 1 vertex or a class 0 vertex (a class 3 vertex is not adjacent to another vertex of degree 3). Hence $\mu^*(v) = 3 - 2(\frac{2}{7}) + \frac{1}{7} = \frac{18}{7}$.

If v is class 3, then by Lemma 3.18 each 3-vertex at distance 2 from v is a class 1 vertex. Hence $\mu^*(v) = 3 - 3(\frac{2}{7}) + 3\frac{1}{7} = \frac{18}{7}$.

Finally, let v be class 1. By Lemma 3.19, v is adjacent to at most one class 2 vertex (and no class 3 vertex). Also, v is distance 2 from at most one class 3 vertex. Hence v gives away $\frac{2}{7}$ to its neighbor of degree 2 and $\frac{1}{7}$ to each of at most two vertices of degree 3. Hence $\mu^*(v) \geq \frac{18}{7}$ unless v is adjacent to one class 2 vertex w, has distance 2 from a class 3 vertex x, and does not receive $\frac{1}{7}$ from its other neighbor y of degree 3. Hence y cannot be class 0; it must be class 1. This leaves us with a Y-configuration, where $v_1 = x$, $v_2 = v$, $v_3 = w$, and $v_4 = y$. By Lemma 3.20, G contains no such configuration. □

Corollary 3.22. *If G is a planar subcubic graph with girth at least 9, then $\chi_l(G^2) \leq 6$.*

Proof: From Lemma 1.2, we see that $Mad(G) < \frac{18}{7}$. By Theorem 3.21, this implies that $\chi_l(G^2) \leq 6$. □

3.2.3 Efficient Algorithms

The proofs of Theorems 3.15 and 3.21 are examples of a large class of discharging arguments that convert easily into linear-time algorithms. The algorithm for each consists of finding a reducible configuration H (7-reducible for Theorem 3.15 and $6'$-reducible for Theorem 3.21), recursively coloring $G^2 \setminus V(H)$, then extending the coloring to G^2. To achieve a linear running time, we need to find the reducible configuration in amortized constant time. We make no effort to discover the optimal coefficient on the linear term in the running time; we only outline the technique to show that the algorithm can be made to run in linear time.

First we decompose G, by removing one reducible configuration after another; when we remove a configuration from G, we add it to a list A (of removed configurations). After decomposing G, we build the graph back up, adding elements of A in the reverse of the order they were removed. When we add back an element of A, we color all of its vertices. In this way, we eventually reach G, with every vertex colored. We call these two stages the decomposing phase and the rebuilding phase. Rebuilding takes constant time per configuration. We must show how to find configurations to remove during the decomposing phase.

Our plan is to maintain a list B of reducible configurations. We begin with a preprocessing phase, in which we store in B every reducible configuration in the original graph. Using brute force, we can do this in linear time, since we have only a constant number of reducible configurations, each configuration has bounded size, and each vertex appear in at most a constant number of reducible configurations.

When we remove a reducible configuration H from G, we may create new reducible configurations. We can search for these new reducible configurations in constant time, since they must be adjacent to H. We add each new reducible configuration to B. In removing H, we may have destroyed one or more reducible configurations in B (for example, they may contain vertices of H). We ignore the destroyed configurations in B. At every point in time, B contains all the reducible configurations in the remaining graph, along with possibly many "destroyed" reducible configurations.

Therefore, when we choose a configuration H from B to remove from the remaining graph, we must verify that H has not been destroyed. If H has been destroyed, then we discard it and proceed to the next configuration in B. We will show that the entire process of decomposing G (and building A) takes linear time. (However, during the process, the time required to find a particular configuration to add to A may not be constant.)

Theorems 3.15 and 3.21 guarantee that as we decompose G, list B will never be empty. Our only concern is that perhaps B may contain "too many" destroyed configurations. It suffices to show that throughout both the preprocessing phase and the decomposing phase, only a linear number of configurations can be added to B. In the original graph G, each vertex can appear in only a constant number of reducible configurations; hence, in the preprocessing phase, only a linear number of reducible configurations are added to B.

During the decomposing phase, if we remove a destroyed configuration from B, we discard it without adding any configurations to B. If we remove a valid configuration from B, we add only a constant number of configurations to B. Each time we remove a valid configuration from B, we decrease the number of vertices in the remaining graph; hence we remove only a linear number of valid configurations from B. Thus, during the decomposing phase, we add only a linear number of configurations to B. As a result, the decomposing phase runs in linear time.

During the rebuilding phase, we use constant time to add a configuration back, and constant time to color the configuration's vertices (we do this using the lemma that proved the configuration was reducible). List A contains only a linear number of configurations, hence, the rebuilding phase runs in linear time.

Since the preprocessing phase, decomposing phase, and rebuilding phase all run in linear time, our complete algorithm runs in linear time.

Chapter 4

(7,2)-edge-choosability of cubic graphs

Question 4.1. *What is the minimum integer r such that if we give lists of size r to the edges of a 3-regular graph, then we can choose sublists of size 2 so that the sublists for incident edges are disjoint?*

A graph is *(r,s)-edge-choosable* if whenever each edge is given a list of r colors, we can choose a sublist of s colors for each edge so that incident edges receive disjoint sublists. Given a graph G and an integer s, it is natural to ask for the minimum r such that G is (r, s)-choosable.

In a Problem of the Month, Bojan Mohar asked what the minimum r is such that every 3-regular graph is $(r, 2)$-edge-choosable. He conjectured that every 3-regular graph is (7,2)-edge-choosable and suggested the Petersen graph as a candidate for a counterexample.

It is not difficult to show that every 3-regular graph is (8,2)-edge-choosable, using a generalization of Brooks' Theorem. Tuza and Voigt [43] proved that: If a connected graph G is not complete and not an odd cycle, then G is $(\Delta(G)m, m)$-vertex-choosable for all $m \geq 1$. Since the line graph of a 3-regular graph has maximum degree 4, every 3-regular graph is (8,2)-edge-choosable.

It is also not difficult to construct a 3-regular graph that is not (6,2)-edge-choosable. Form G by subdividing an edge of K_4. We see by inspection that G is not (6,2)-edge-colorable and thus is not (6,2)-edge-choosable. Hence, any 3-regular graph that contains G is not (6,2)-edge-choosable. As a result, the conjecture that every 3-regular graph is (7,2)-edge-choosable is sharp if true.

In this section, we show that every 3-edge-colorable graph is (7,2)-edge-choosable and that the Petersen graph is (7,2)-edge-choosable. Ellingham and Goddyn [12] showed that planar d-regular d-edge-colorable multigraphs are d-edge-choosable (thus planar cubic graphs are (6,2)-edge-choosable). Recently, Haxell and Naserasr [24] showed that the Petersen graph is (6,2)-edge-choosable. Showing that lists of size only 6 suffice is a stronger result. However, both papers use the Alon-Tarsi Theorem and thus provide only existence proofs. Here we give a simple algorithm for choosing the colorings from lists of size 7.

Each edge of a 3-regular graph is incident to four other edges. Thus, we could have as many as eight restrictions on the colors we choose for an edge. Our main idea is to show that we can choose colors for two of these incident edges, while only increasing the number of relevant restrictions by one.

4.1 The Key Lemma

Our main lemma is a generalization of the well-known result [42] that even cycles are $(2m, m)$-edge-choosable. To understand the following proof, it may be useful to consider the case when G is an even cycle. In general, however, B need not be a matching.

Lemma 4.2. *Let $A = \{a_1, a_2, \ldots, a_k\}$ be a matching and $B = \{b_1, b_2, \ldots, b_k\}$ be an edge set such that b_i is incident to a_i and a_{i+1}, but not incident to any other edge in A (the subscript indices are viewed modulo k). Let the list assigned to edge e be $L(e)$, with all the lists having the same size. It is possible to choose one color for each edge of A from its list so that at most one color in $L(b_i)$ is used on a_i and a_{i+1}.*

Proof: We will choose a color $c(e)$ for each edge e in $A \cup B$.

If the lists for all edges n $A \cup B$ are identical, then use the same color on each edge of A. If the lists are not all identical, then lists differ for two successive edges in the alternating list $a_1, b_1, a_2, b_2, \ldots a_n, b_n$. We may assume that these are a_1 and b_n.

Choose $c(a_1) \notin L(b_n)$. If $c(a_1) \notin L(b_1)$, then we have the freedom to choose $c(a_2)$ from $L(a_2)$ arbitrarily. If $c(a_1) \in L(b_1) \cap L(a_2)$, then let $c(a_2) = c(a_1)$. Finally, if $c(a_1) \in L(b_1) - L(a_2)$, then we can choose $c(a_2) \in L(a_2) - L(b_1)$. In each case, at most one of the colors chosen for the edges a_1 and a_2 incident to b_1 is in $L(b_1)$.

Continue in the same manner choosing colors for edges $a_3, a_4, \ldots a_n$, so that at most one color from $L(b_i)$ is used on a_i and a_{i+1}. Finally, also b_n has at most one color prohibited, since $c(a_1) \notin L(b_n)$. $\qquad \square$

Corollary 4.3. *[42] Even cycles are $(2m, m)$-edge-choosable.*

Proof: Partition the edges of the cycle into two matchings, A and B. Simultaneously choose one color for each edge of A as guaranteed by Lemma 5.5. Repeat this step m times. (Each time we repeat this step, it may be necessary to restrict the size of some lists by 1, so that all lists have equal sizes.) At this point, each edge of B has at least m available colors. $\qquad \square$

It is not immediately obvious that Lemma 5.5 can be used to prove anything more than Corollary 4.3. Its power lies in choosing the edge sets A and B cleverly.

Theorem 4.4. *Graphs that are 3-edge-colorable are (7,2)-edge-choosable.*

Proof: Let G be a 3-edge-colorable graph. It is straightforward to add edges and vertices to G to form a cubic graph that is 3-edge-colorable. We may thus assume that G is cubic and 3-edge-colorable, since every subgraph of a (7,2)-edge-choosable graph is (7,2)-edge-choosable.

Let J, K, and L be the three color classes that partition $E(G)$. Since the graph is cubic, the color classes have the same size. We apply Lemma 5.5 twice. First apply Lemma 5.5 with J as A and K as B. At this point, we have choosen one color for each edge of J. The lists of colors remaining available have size at least 6 for edges of J and K and size at least 5 for edges of L. Now apply Lemma 5.5 with J as A and L as B. (Since the lemma requires that all list sizes be the same, arbitrarily restrict the lists on edges in J and L to sets of size 5 before applying the lemma.) After doing this, we have chosen two colors for each edge in J

but no colors for any edge in K or L. Each edge of K and L has at least four colors remaining available in its list. Since the edges of K and L form vertex-disjoint even cycles, we may apply Lemma 4.3 to each cycle to complete the selection of the coloring. □

We can use the ideas in Theorem 4.4 to prove that many other graphs are (7,2)-edge-choosable.

A *snark* is a bridgeless 3-regular graph with edge-chromatic number 4, girth at least 5, and cyclic-connectivity at least 4. Snarks first became of interest because the 4 Color Theorem is equivalent to the fact that there are no planar snarks. They remain of interest because many important conjectures are known to be true if and only if they are true for snarks.

Theorem 4.5. *If a 3-regular graph G has girth at least 5 and has an edge uv such that $G \setminus \{u,v\}$ is Hamiltonian, then G is (7,2)-edge-choosable.*

Proof: The ideas in this proof are similar to those used to prove that 3-edge-colorable graphs are (7,2)-edge-choosable. To understand the proof, it may be convenient to consider the Petersen graph (shown below), which is the simplest class 2 graph to which the theorem applies.

Let $a_1, a_2, \ldots, a_{n-2}$ denote the edges of the Hamiltonian cycle in $G \setminus \{u,v\}$. Let b_i denote the edge that is incident to a_i and a_{i+1}. Note that often $b_i = b_j$ for $i \neq j$; in fact, each edge not on the Hamiltonian cycle is labeled as two distinct b_i except for edge uv and the four edges incident to uv. Let p and q be the subscripts of the b_is incident to vertex u and let r and s be the subscripts of the b_is incident to vertex v; assume that $p < q$ and $r < s$ (also assume $q < s$).

If $L(a_i) = L(b_i) = L(a_{i+1})$ for all i, then by transitivity, the lists on all edges are the same. In that case we can use colors 1 and 2 on all the b_is; Then we are done, since the Hamiltonian cycle has lists of size 5 and is (4,2)-edge-choosable. So assume that not all the lists are the same.

We may assume that an pair of incident edges a_i and b_i have distinct lists; by renaming the edges, we may assume that $L(a_1) \neq L(b_1)$. Choose $c(b_1) \notin L(a_1)$; remove $c(b_1)$ from $L(a_2)$ (if $c(b_1) \notin L(a_2)$, then remove an arbitrary color from $L(a_2)$). We also must remove $c(b_1)$ from the lists on other edges incident to edge b_2; so if edge b_2 is also edge b_j, then we also remove $c(b_1)$ from $L(a_j)$ and $L(a_{j+1})$. Now since $|L(b_2)| = 7$ and $|L(a_2)| = 6$, we can choose $c(b_2) \notin L(a_2)$. After we remove $c(b_2)$ from $L(a_3)$, we have $|L(b_3)| > |L(a_3)|$. Continuing like this, we always have $|L(b_i)| > |L(a_i)|$ at the time we choose $c(b_i)$. The two possible exceptions to this are $i = q$ and $i = s$; this is because our choice for $c(b_p)$ reduced the size of $L(b_q)$ but not the size of $L(a_q)$, similarly $c(b_r)$ reduced $|L(b_s)|$ but not $|L(a_s)|$. When we need to choose $c(b_q)$, we do so arbitarily (we will save a color on edge a_q later). When we need to choose $c(b_s)$, we choose arbitarily from $L(b_s) \setminus L(uv)$ (this is possible because at this point $|L(b_s)| = 6$ and $|L(uv)| = 4$, since we have removed $c(b_p)$, $c(b_q)$, and $c(b_r)$ from $L(uv)$). After we have chosen one color for each edge b_i, we choose a second color each for b_q and b_s. Since G has girth at least 5, note that $(q-1) \notin \{p,r,s\}$; thus, we have already chosen two colors for edge b_{q-1}. This means that $|L(a_q)| = 4$, but $|L(b_q)| = 5$; so we can choose a second color for b_q, this one from $L(b_q) \setminus L(a_q)$. Similarly, we can choose a second color for b_s, this one from $L(b_s) \setminus L(b_r)$. At this point, $|L(uv)| = 2$, so we choose two colors for edge uv. Next, we choose a second color for each of b_p and b_r. Each edge on the Hamiltonian cycle has four colors available, so we can finish the coloring since even cycles are (4,2)-edge-choosable. □

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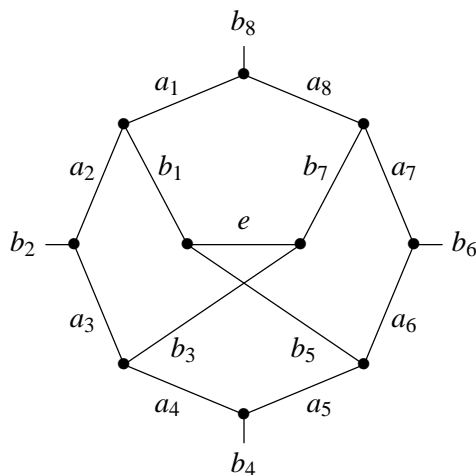
Figure 4.1. The Petersen Graph on the torus with one crossing.

Corollary 4.6. *All snarks on at most 24 vertices are (7,2)-edge-choosable; so are the double star snark, the Szekeres snark, the Goldberg snark, the Watkins snarks of orders 42 and 50, and all cyclically 5-edge-connected snarks of order 26.*

Proof: Cavicchioli et al. give drawings of all snarks on at most 24 vertices (there are 67 such non-isomorphic snarks) as well as all cyclically 5-edge-connected snarks of order 26. Their drawings are designed to illustrate that the snarks are almost Hamiltonian, but they also make it easy to see that each snark satisfies the hypotheses of Theorem 4.5. Each of the other snarks listed in the theorem also has an edge uv such that $G \setminus \{u,v\}$ is Hamiltonian. □

In fact, we do not know of any snarks that do not satisfy the hypotheses of Theorem 4.5. This leads us to conjecture the following.

Conjecture 4.7. *Every snark G has some edge uv such that $G \setminus \{u,v\}$ is Hamiltonian.*

If true, this conjecture implies that all snarks are $(7,2)$-edge-choosable.

4.2 2-sum-chromatic number

We would like to show that many more 3-regular class 2 graphs are $(7,2)$-edge-choosable. Our plan is to generalize the technique we used to show that 3-regular class 1 graphs are $(7,2)$-edge-choosable. Initially, we consider 3-regular graphs with a 1-factor M. We will choose colors for the edges of M so that we save colors on the edges in the 2-factor $E(G) \setminus M$. As in the proof of Theorem 4.4, we will aim to save an average of one color per edge of the 2-factor. Since G is not class 1, the 2-factor will contain odd cycles. It is easy to see that odd cycles are not $(4,2)$-edge-choosable (since they are not $(4,2)$-edge-colorable). Thus, we consider list size assignments f for the cycle C_k, such that C_k is $(f,2)$-edge-choosable and the sum of the list sizes is $4k$. Since the line graph of a cycle is isomorphic to the cycle, we assign lists to and choose colors for the vertices of the cycle (rather than the edges). For a graph G, we write $\chi_{sc}(G,2)$ to denote the minimum sum of list sizes f such that G is $(f,2)$-choosable; we call this the 2-sum-chromatic number of G.

Lemma 4.8. *The 2-sum-chromatic number of P_k is $4k-2$.*

Proof: To prove the upper bound, let $f(v_1) = 2$ and $f(v_i) = 4$ for all $i > 1$. We greedily choose colors for the vertices in order of increasing subscript. We restate the lower bound as follows. Given a list size assignment f on a path P_k, if the sum of the list sizes is at most $4k-3$, then P_k is not $(f,2)$-choosable. We prove the lower bound by induction on the number of vertices in the path with list sizes either 2, 5, or 6. For the base case, assume that no vertices have list sizes 2, 5, or 6; thus, at least three vertices have list size 3.

Suppose that two successive vertices with list size 3 are separated by an even number of vertices (say $2l$) with list size 4, then we assign the list $\{a,b,c\}$ to the vertices with list size 3 and the list $\{a,b,c,d\}$ to the vertices with list size 4. Each of the colors a, b, and c can appear on at most $l+1$ of these vertices and color d can appear on at most l vertices; since $3(l+1)+1 < 2(2l+2)$, the path is not $(f,2)$-chooable.

Suppose instead that each pair of successive vertices with list size 3 are separated by an odd number of vertices with list size 4. Let u_1, u_2, and u_3 be the vertices with list size 3. Assign to u_1, u_2, and u_3 the lists $\{a,b,c\}$, $\{b,c,d\}$ and $\{a,c,d\}$ respectively; assign the list $\{a,b,c,d\}$ to all other vertices (say there are $2l$ of them). Colors a, b, and d can each appear on at most $l+1$ vertices; color c can appear on at most $l+2$ vertices. Since $3(l+1)+l+2 < 2(l+3)$, the path is not $(f,2)$-choosable.

Now consider the inductive step. If some vertex u has list size 6, then we consider the two paths P_{k_1} and P_{k_2} that result by deleting u. Note that $6 + (4k_1 - 2) + (4k_2 - 2) = 4(k_1 + k_2 + 1) - 2 = 4k - 2$; so either P_{k_1} has list size sum at most $4k_1 - 3$ or P_{k_2} has list size sum at most $4k_2 - 3$.

If some vertex u has list size 2, again we consider the two paths P_{k-1} and P_{k_2} that result by deleting u. Now however, we decrease by 2 the list size of each neighbor of u. Note that $2 + ((4k_1 - 2) + 2) + ((4k_2 - 2) + 2) = 4(k_1 + k_2 + 1) - 2$; so again, one of the resulting paths P_m must have list sum size at most $4m - 3$.

Suppose instead that every vertex has list sum size 3, 4, or 5. By the pigeonhole principle, two vertices with list size 3 must be separated only by vertices with list size 4; call these vertices u_1 and u_2. By the argument above, these vertices must be separated by an odd number of vertices with list size 4. We assign the list $\{a,b,c\}$ to u_1 and the list $\{a,b,d\}$ to u_2, and we assign the list $\{a,b,c,d\}$ to all other vertices. Observe that the only valid coloring from this list uses colors a and b on both u_1 and u_2. Contract the path from u_1 to u_2 down to a single vertex and assign it the list $\{a,b\}$. Thus, the original path is only $(f,2)$-choosable if this new path is $(f',2)$-choosable. Hence, we induct as above when a vertex had a list size of 2. $\qquad\square$

The proof of the lower bound in the previous lemma allows us to give a linear-time algorithm to determine if a path P_k is $(f,2)$-choosable whenever the list size sum of f is $4k-2$. When the path is not $(f,2)$-choosable, our algorithm constructs a list assignment L such that P_k is not $(L,2)$-colorable and also gives a proof of this fact.

Theorem 4.9. *Given a path P_k and a list size function f such that the list size sum is $4k-2$, we can determine in linear time whether P_k is $(f,2)$-choosable.*

Proof: Our proof follows closely the proof of the lower bound in the previous lemma.

If a vertex v has list size 6, then P_k is $(f,2)$-choosable if and only if each of the paths formed by deleting v from P_k are $(f,2)$-choosable. Suppose the paths have lengths k_1 and k_2. By Lemma 4.8, we know that

the paths must have list size sums at least $4k_1 - 2$ and $4k_2 - 2$, respectively. Since $4k_1 - 2 + 4k_2 - 2 + 6 = 4(k_1 + k_2 + 1) - 2$, we see that these bounds each hold with equality; so we proceed by induction.

If a vertex v has list size 2, we assign it the colors a and b, and also assign these colors to each of its neighbors. The path P_k is $(f, 2)$-choosable if and only if each of the paths formed by deleting v from P_k are $(f, 2)$-choosable when the list size of each neighbor of v is decreased by 2; again we proceed by induction.

If each vertex of P_k has list size 3, 4, or 5, then by the pigeonhole principle, two vertices u_1 and u_2 each have list size 3 and the list size of each vertex between them is 4 (and there are an odd number of vertices between them); denote the vertices between u_1 and u_2 by $v_1, v_2, \ldots, v_{2l+1}$. In the proof of Lemma 4.8, we showed how to assign lists so that the only valid coloring uses the colors a and b on both u_1 and u_2. In that case, the path is $(f, 2)$-choosable only if each of the paths formed by deleting vertices u_1 through u_2 (inclusive) are $(f, 2)$-choosable when we decrease by 2 the list size of the neighbors of u_1 and u_2. Given any list L, we can choose a color c_1 for v_1 that is not present at u_1; delete c_1 from the list at v_2. Now we can choose a color for v_3 that is not present at v_2, etc. Eventually, we choose a color for vertex v_{2l+1} that is not present at v_{2l}; however, this reduces the list size at u_2 to two, so we choose two colors for u_2. This in turn reduces the list size for v_{2l+1} to one. Working backwards from the greatest index to the least, we see that all the remaining choices are forced, but that the path between u_1 and u_2 is indeed $(f, 2)$-choosable. $\square$

We are more interested in the 2-sum-chromatic number of the cycle than of the path; however, Lemma 4.8 is useful, since it gives the lower bound of $4k - 2$ on the 2-sum-chromatic number of the cycle C_k. It is easy to prove an upper bound of of $4k$ on the 2-sum-chromatic number of C_k; so we want to improve the lower bound. We introduce the idea of run to improve this lower bound.

Given a list assignment L, we define a *run* to be a maximal path (or cycle) such that each vertex has a common color. If a path P_k is a run for color c, then at most $\lceil k/2 \rceil$ vertices of P_k can receive color c; if a color c appears at each vertex of C_k, then at most $\lfloor n/2 \rfloor$ vertices of C_k can receive color c. For a list assignment L, let $s(L)$ denote the sum over all runs of the maximum number of vertices in the run that can receive the color of the run.

Lemma 4.10. *Given a list assignment L, if $s(L)$ is less than $2n$, then the cycle C_n is not $(L, 2)$-colorable.*

Proof: If a cycle C_n is $(L, 2)$-colorable, then in total the n vertices receive $2n$ colors. If the sum over all runs of the maximum number of vertices in the run that can receive the color is less than $2n$, then C_n is not $(L, 2)$-colorable. $\square$

Lemma 4.11. *Let C_n be an odd cycle. Let L be a list assignment. If the same two colors appear at each vertex of C_n and each other run has even length, then C_n is not $(L, 2)$-colorable.*

Proof: This is a direct application of Lemma 4.10. The maximum number of vertices that can receive one of the two colors that appears everywhere is $2\lfloor n/2 \rfloor = n - 1$. Since each other run has even length, the sums of the contributions of all the other runs will be n. Thus, the n vertices can be assigned at most $2n - 1$ colors. Hence, C_n is not $(L, 2)$-colorable. $\square$

Lemma 4.12. *If a cycle C_k is $(f, 2)$-choosable and the list size sum of f is less than $4k$, then each vertex receives a list of size 3, 4, or 5; furthermore, between each two vertices with lists of size 3 some vertex has list size 5.*

Proof: Suppose that C_k is $(f,2)$-choosable and the list size sum of f is at most $4k-1$ and that some vertex v receives a list size of 6. The cycle C_k is $(f,2)$-choosable only if the path P_{k-1} formed by deleting v is $(f,2)$-choosable; however, the list size sum of f for this path is at most $4(k-1)-3$. By Lemma 4.8, the path P_{k-1} is not $(f,2)$-choosable, so the cycle C_k is not $(f,2)$-choosable. Suppose instead that a vertex v receives a list of size 2; the argument is analgous. We delete v and decrease by 2 the list size of each of its neighbors. Again, the resulting path is not $(f,2)$-choosable, so neither is the original cycle.

If there exist two vertices u_1 and u_2 with lists of size 3 and all vertices between them have lists of size 4, we assign lists in the following way. If the number of vertices between u_1 and u_2 is even, assign $\{a,b,c\}$ to u_1 and u_2 and assign $\{a,b,c,d\}$ to each vertex between them; by Lemma 4.10, the path from u_1 to u_2 is not $(L,2)$-colorable. If the number of vertices between u_1 and u_2 is odd, assign $\{a,b,c\}$ to u_1, assign $\{a,b,d\}$ to u_2 and assign $\{a,b,c,d\}$ to each vertex between them; it is straightforward to verify that the only valid color of the path from u_1 to u_2 uses colors a and b on both vertex u_1 and vertex u_2. Thus, we can proceed as if the path from u_1 to u_2 were replaced by a single vertex with list $\{a,b\}$. Above, we showed that the cycle is not $(f,2)$-choosable for such a function f. $\qquad\square$

From the previous lemma, we conclude that if C_k is $(f,2)$-choosable for a list size function f with list size sum less than $4k$, then f assigns each vertex a list size of 4 except for possibly one vertex, which may receive a list size of 3. If k is even, the cycle C_k is $(f,2)$-choosable for such a function f; we omit the proof, since the result will be of little use to us in the application to $(7,2)$-edge-choosability. Since an odd cycle is not $(4,2)$-colorable, it is easy to see that the for odd k, $\chi_{sc}(C_k) = 4k$. Now we consider for which f with list size sum $4k$ the cycle C_k is $(f,2)$-choosable.

Lemma 4.13. *Suppose we are given a cycle C_k and a list size function f such that f has list size sum $4k$. If any vertex has list size 2 or 6, or if two vertices have list size 3 and each vertex between them has list size 4, then we can determine whether C_k is $(f,2)$-choosable in linear time.*

Proof: If a vertex u has list size 2 or 6, we delete vertex u and proceed as in the proof of Theorem 4.9. If vertices u_1 and u_2 each have list size 3 and all vertices between them have list size 4, we delete the path from u_1 to u_2 and proceed as in Theorem 4.9. $\qquad\square$

For every list size assignment f with list size sum $4k$ we can determine in linear-time the $(f,2)$-choosability of a cycle C_k unless each list size is 3, 4, or 5 and the vertices with list sizes 3 and 5 alternate as we proceed around the cycle. We call such a function f *well-formed*. Our next few lemmas and theorems work toward a linear-time algorithm to determine whether a cycle is $(f,2)$-choosable given a well-formed function f.

Since the 3s and 5s must alternate, it is convenient to view f as a series of blocks, where a *block* is a 3, followed by 0 or more 4s (say a 4s), followed by a 5, followed by 0 or more 4s (say b 4s); frequently we include the initial 3 of the next block, but we never choose colors for it or decrease its list. We are mainly concerned with the parity of a and b; thus, we consider four types of blocks: odd/odd, even/even, odd/even, and even/odd. We use the term *string* to denote the list sizes of one or more adjacent vertices that are not of the form of one of these blocks. We also need a means to denote that one or two colors have already been chosen for a vertex; thus, if a vertex has a remaining list of size c and already has one or two colors chosen,

we denote the vertex by $\dot{c}$ or $\ddot{c}$, respectively. We begin by proving that for a certain class of functions f, the cycle C_k is not $(f,2)$-choosable; then we prove that for all well-formed functions beside this class, the cycle C_k is $(f,2)$-choosable.

Theorem 4.14. *If C_n is an odd cycle and f consists only of odd/odd blocks and odd/even blocks, then C_n is not $(f,2)$-choosable.*

Proof: Given the list size for each vertex, we construct a list L such that C_n is not $(L,2)$-colorable. We construct one list for an odd/odd block and another list for an odd/even block. Finally, we show that if each block of the cycle receives the list of the appropriate type, then C_n is not $(L,2)$-colorable. To prove this, we use Lemma 4.10.

Suppose that an odd/odd block consists of one 3, r 4s, one 5, s 4s (where r and s are odd); we assign the following lists: the first 3 receives $\{a,b,c\}$, the first r 4s receive $\{a,b,c,d\}$, the 5 receives $\{a,b,c,d,e\}$, and the last s 4s receive $\{a,b,c,e\}$. Suppose that an odd/even block consists of one 3, t 4s, one 5, u 4s (where t is odd and u is even); we assign the following lists: the first 3 receives $\{a,b,c\}$, the first t 4s receive $\{a,b,d,e\}$, the 5 receives $\{a,b,c,d,e\}$, and the last u 4s receive $\{a,b,c,e\}$.

To apply Lemma 4.10, we must verify two facts: 1) some two colors appear at each vertex of C_n and s) every other run has even length. It is easy to see that colors a and b appear at every vertex. Every run of color e is of length $1+r$ or length $1+t$; since r and t are odd, these runs have even length. Every run of color 4 is of length $1+s$ or length $t+1+u$; these lengths are always even. Note that every run of color c starts at an odd/even block and ends at an odd/even block. Suppose that a run of color c has l odd/odd blocks between its starting and ending block. The length of the run is $1+d+l(1+a+1+b)+1$; this length is even. Thus, we can apply Lemma 4.10; so the theorem holds. $\square$

Corollary 4.15. *If C_n is an odd cycle and f consists only of odd/odd blocks and even/odd blocks, then C_n is not $(f,2)$-choosable.*

Proof: If we relabel the vertices of C_n in counterclockwise order (rather than clockwise), then the even/odd blocks become odd/even blocks (and odd/odd blocks remain odd/odd blocks); hence, the result follows immediately from Theorem 4.14. $\square$

In fact, the list size assignment functions in Theorem 4.14 and Corollary 4.15 are the only well-formed list assignment functions f such that a cycle C_n is not $(f,2)$-choosable. We prove this in the next theorem.

Theorem 4.16. *If f is a well-formed list size assignment function and f does not meet the hypothesis of Corollary 4.15, then C_n is $(f,2)$-choosable.*

Proof: A *good string* is a string that begins with $3\dot{2}$ and ends with a disjoint $\dot{2}3$ such that when the lists of the first and last $\dot{2}$ vertices are each reduced by one color we can successfully choose all the necessary colors for all the vertices between them.

Claim 1. We can transform a block of the form $34(44)^*5(44)^*43$ (odd/odd) into a string of the form $3(\dot{3}3)^*$.

Let $v_1, v_2, \ldots, v_{2k+1}$ be the first path of vertices with lists of size 4. Let v_0 be the initial vertex with list size 3 and let v_{2k+2} be the vertex with list size 5. Since $f(v_1) > f(v_0)$, we can choose a color c_1 for v_1 that does not appear at v_0. We may assume that c_1 appears at vertex v_2, and hence using c_1 at v_2 reduces the number of colors available at v_2. Now, since $f(v_3) > f(v_2) - 1$, we can choose a color for v_3 that does not further reduce the number of colors at v_2. Proceeding in this manner, we eventually reach a block of the form $3\dot{3}(3\dot{3})^*4(44)^*43$. In the same way, we begin at the end of the block and work toward the middle-choosing a color for the next to last vertex that does not appear at the last vertex, etc. Ultimately, we reach a block of the form $3(\dot{3}3)^*$.

Claim 2. We can transform a block of the form $3(44)^*5(44)^*3$ (even/even) into a good string of the form $3(\dot{2}\dot{2})^*4(\dot{2}\dot{2})^*3$.

We begin the same way as in Claim 1, by choosing a color for vertex v_1 that does not appear at v_0. Eventually, we reach the form $3(\dot{3}3)^*5(3\dot{3})^*3$. Now we choose one color each for the two neighbors of the vertex with list size 5 such that its list is reduced by at most one; this gives the form $3(\dot{3}3)^*\dot{2}\dot{2}4\dot{2}\dot{2}(3\dot{3})^*3$. Now we work out from the center vertex; at each step we choose a color for a vertex with no color chosen, so that the list of colors on its neighbor nearer the center does not decrease. Eventually, we reach $3(\dot{2}\dot{2})^*4(\dot{2}\dot{2})^*3$. If the lists on the initial and final $\dot{2}$ are each reduced by one color, we color greedily from both the start and end of the string; thus, this string is good.

Claim 3. We can transform a string of the form $34(44)^*5(44)^*3(44)^*54(44)^*3$ (an odd/even block followed by an even/odd block) into a good string of the form $3\dot{2}^*\dot{3}(3\dot{3})^*\dot{2}^*3$.

We choose a color for each neighbor of a vertex with list size 3 so that none of the lists of size 3 are reduced; moving away from the lists of size 3 (as in Claims 1 and 2), we eventually reach a string of the form $3\dot{3}(3\dot{3})^*43(\dot{3}3)^*4(\dot{3}3)^*\dot{3}3$. Now we choose a color for the first vertex with list size 4 that does not appear on the neighbor that follows it; similarly, we choose a color for the last vertex with list size 4 that does not appear on the neighbor that preceeds it. This gives us a string of the form $(3\dot{3})^*3\dot{2}\dot{3}3(3\dot{3})^*\dot{3}2\dot{3}(3\dot{3})^*$. Again we choose a color for the neighbor of the first vertex with list size 2 that does not appear on that vertex; similarly for the last vertex with list size 2. Proceeding outward, we reach the form $3\dot{2}(\dot{2}\dot{2})^*\dot{3}3(3\dot{3})^*\dot{3}\dot{2}(\dot{2}\dot{2})^*3$.

If the lists on the initial and final $\dot{2}$ are each reduced by one color, we color greedily from both the start and end of the string; this gives us the form $\ddot{0}^*\dot{2}3(3\dot{3})^*\dot{2}\ddot{0}^*$. By repeatedly choosing a color for the first vertex with list size 3 that does not appear on its preceeding neighbor, we reach the form $\ddot{0}^*\dot{2}\dot{2}^*3\dot{2}\ddot{0}^*$. Now we choose a color for each neighbor of the vertex with list size 3 so that we reduce its list by at most 1; then we greedily finish the coloring. Hence, the string $3\dot{2}(\dot{2}\dot{2})^*\dot{3}3(3\dot{3})^*\dot{3}\dot{2}(\dot{2}\dot{2})^*3$ is good.

Claim 4. We can transform a string of the form $3(44)^*54(44)^*3(44)^*45(44)^*3$ (an even/odd block followed by an odd/even block) into a good string of the form $3\dot{2}^*\ddot{0}\ddot{0}^*\dot{2}^*3$. We begin in the standard way, choosing a color for each neighbor of a 3 that does not appear on that 3; working away from the 3s, we eventually reach $3(\dot{3}3)^*4\dot{3}(3\dot{3})^*3\dot{3}(3\dot{3})^*4(3\dot{3})^*3$. Now we choose a color for the first 4 that does not appear on its preceeding neighbor; working to the right, we reach the following form $3(\dot{3}3)^*3\dot{2}(\dot{2}\dot{2})^*\dot{2}\dot{2}(\dot{2}\dot{2})^*4(3\dot{3})^*3$. Now we choose a color for the last 4 that does not appear on its succeeding neighbor; working to the left, we reach the following form $3(\dot{3}3)^*\dot{2}\ddot{0}\ddot{0}*\dot{2}(3\dot{3})^*3$. Finally, we choose a color for each neighbor of a $\dot{2}$ that do not appear on the $\dot{2}$; working outward, we reach the form $3\dot{2}^*\ddot{0}\ddot{0}^*\dot{2}^*3$. It is easy to see that this is a good string. If the lists on the initial and final $\dot{2}$ are each reduced by one color, we color greedily from both the start and end of the string; thus, this string is good.

Claim 5. If one or more odd/odd blocks are adjacent to any of the strings in Claim 2, Claim 3, or Claim 4 (or inserted between the odd/even and even/odd blocks of the strings in Claim 3 or Claim 4) the adjacent odd/odd blocks can be absorbed during the transformation; that is, the odd/odd blocks become part of the resulting good string.

We first transform the odd/odd block to a string of the form $3(\dot{3}3)^*$. The key realization is that during the transformation processes for the strings in Claims 2, 3, and 4, we pass through a pattern that includes a string of the form $(\dot{3}3)^*$ adjacent to the transformed odd/odd block; from this point on, we continue the transformation as though the transformed odd/odd block is part of the block being transformed.

Claim 6. If a block of the form $34(44)^*5(44)^*3$ (an odd/even block) follows immediately after a good string, then the odd/even block can be absorbed into the preceeding good string.

By working from the outer 3s in toward the 5, we reach the string $3\dot{3}(3\dot{3})^*4(3\dot{3})^*3$. Choose a color for the 4 that does not appear on its succeeding neighbor, then work back to the left iteratively choosing a color for a vertex with no colors choosen that does not appear on its succeeding neighbor; this yields the form $3\dot{2}^*\dot{3}(3\dot{3})^*3$. Choose a color for the first 3 that does not appear on the $\dot{2}$ that preceeds it (from the preceeding good string). This reduces its succeeding neighbor to a $\dot{1}$; now we proceed greedily to the right. This yields the form $\ddot{0}\ddot{0}^*\dot{2}\dot{2}^*3$; it is easy to see that this is a good string.

Claim 7. We can first transform (or absorb) all blocks into good strings; then we can finish choosing 2 colors for each vertex.

If L contains any odd/even or even/odd block, we assume without loss of generality that L contains odd/even blocks (and possibly also even/odd blocks). Iteratively, we apply Claims 1, 2, 3, 4, 5, and 6; our goal is to transform (or absorb) every block into a good string. Note that by Claim 6 (and symmetry), if an even/odd block immediately preceeds a good string, the block can be absorbed into the string. Suppose that after all possible transformations (and absorbtions), L contains at least one good string and at least one block that is untransformed. Let B be the first block following a good string that cannot be transformed (or absorbed) into a good string; B must be an even/odd block. As we proceed around the cycle from B, we must only encounter even/odd blocks and transformed odd/odd blocks (for otherwise we could transform B into a good string); however, eventually we reach a good string. Thus, we can absorb the block prior to the good string into the good string. This contradicts our assumption that we had made all possible transformations and absorbtions.

Suppose instead that after all possible transformations (and absorptions), no good string exists; we conclude that L consists entirely of odd/odd blocks and odd/even blocks, which was prohibited by hypothesis.

Now we assume that L has been transformed into a series of good strings. Let v be a vertex with list size 3 between two good strings; let u_1 be the last vertex of the preceeding good string and let u_2 be the first vertex of the succeeding good string. Choose a color for v that does not appear on vertex u_1. Let w_1 be the last vertex at the end of the good string containing u_2; let x and w_2 be the vertices following w_1. Choose a color for x that does not appear on w_2. We assume that the colors we picked for v and x reduced the lists of colors available on u_2 and w_1, respectively. By the definition of good block, we can finish choosing colors for all of the vertices between u_2 and w_1. In fact, when we choose colors for all of these vertices, the second colors for vertices v and x are forced; this in turn forces the final colors for vertices u_1 and w_2. By repeating this process at each vertex between good strings, we eventually choose colors for the entire cycle. $\square$

Theorem 4.17. *Given a list assignment f with list size sum $4n$, the cycle C_n is $(f,2)$-choosable unless there exists a list assignment L that satisfies f such that either $s(L)$ is less than $2n$ or when L is restricted to some path P_k the sum $s(L)$ (restricted to P_k) is less than $2k$.*

Proof: In each case that we proved C_n is not $(f,2)$-choosable for a given list size assignment function f we constructed a list assignment L, such that C_n is not $(L,2)$-colorable. To prove the present theorem, we must simply verify that each such L satisfies the present hypothesis; since this process is straightforward (but tedious), we omit the details. □

In fact, we believe this theorem can be generalized significantly; we end with the following conjecture.

Conjecture 4.18. *Given a list assignment f with list size sum $2ln$, the cycle C_n is (f,l)-choosable unless there exists a list assignment L that satisfies f such that either $s(L)$ is less than ln or when L is restricted to some path P_k the sum $s(L)$ (restricted to P_k) is less than $2lk$.*

It is straightforward to verify that Conjecture 4.18 holds when $l = 1$; Theorem 4.17 proves the case $l = 2$.

Chapter 5

Antimagic Labeling

Problems in graph labeling differ from problems in graph coloring in two important ways. First, "labeling" usually means that the function on the elements receiving labels is injective. Given this, the labels are automatically distinct, and the normal coloring constraints are replaced by relationships among the labels. This leads to the second difference, which is that the constraints involve arithmetic computations with numerical values of the labels.

The most famous graph labeling problem may be the "Graceful Tree Conjecture". Here the vertices of an n-vertex tree must be assigned the labels 1 through n so that the $n-1$ differences between labels at adjacent vertices are the numbers 1 through n. In 1964, Kotzig conjectured that every tree has such a labeling, which later came to be known as a *graceful labeling*. Many other problems of vertex labeling have been introduced over the years; all seem to be quite difficult. Gallian [17] maintains a dynamic survey of results on graph labeling problems; as of 2007, it has more than 800 references.

In this chapter, we study a problem of edge-labeling. For convenience, then, we formally define a *labeling* of a graph G to be a bijection from $E(G)$ to the set $\{1,\ldots,|E(G)|\}$. A *vertex-sum* for a labeling is the sum of the labels on edges incident to a vertex v; we also call this the *sum at v*. A labeling is *antimagic* if the vertex-sums are pairwise distinct. A graph is *antimagic* if it has an antimagic labeling.

The term "antimagic" is motivated by the use of "magic" to describe a labeling whose vertex-sums are identical (strictly speaking, "magic" requires only distinct positive integer labels, not necessarily the consecutive smallest ones). This term in turn arises from the ancient notion of a "magic square", in which numbers are entered in a square grid so that the sums in each row, each column, and each main diagonal are the same. Magic labelings were introduced by Sedláček in 1963. Gallian's survey also presents the known results on magic and antimagic labelings. Most of the results establish that various special families of graphs have various types of magic or antimagic labelings.

Hartsfield and Ringel [20] introduced antimagic labelings in 1990 and conjectured that every connected graph other than K_2 is antimagic. The most significant progress on this problem is a result of Alon, Kaplan, Lev, Roditty, and Yuster [1], which states the existence of a constant c such that if G is an n-vertex graph with $\delta(G) \geq c\log n$), then G is antimagic. Large degrees satisfy a natural intuition: the more edges are present, the more flexibility there is to arrange the labels and possibly obtain an antimagic labeling.

Alon et al. also proved that G is antimagic when $\Delta(G) \geq |V(G)| - 2$, and they proved that all complete multipartite graphs (other than K_2) are antimagic. Hartsfield and Ringel proved that paths, cycles, wheels, and complete graphs are antimagic. Gallian's survey lists no other results on antimagic labelings as such; other work studies other variations of the concept, labeling with additional constraints, etc.

In this chapter, we show that every regular bipartite graph (with degree at least 2) is antimagic. Our proof relies heavily on the Marriage Theorem, which states that every regular bipartite graph has a 1-factor; see Chapter 1. By induction on the vertex degree, it follows that a regular bipartite graph decomposes into 1-factors. Recall that a k-factor is a k-regular spanning subgraph, so the union of any k 1-factors is a k-factor. Throughout this chapter, we refer to the partite sets of the given bipartite graph as A and B, each having size n.

With respect to a given labeling, two vertices *conflict* if they have the same sum. We view the process of constructing an antimagic labeling as resolving the "potential conflict" for every pair of vertices. We will label the edges in phases. When we have labeled a subset of the edges, we call the resulting sum at each vertex a *partial sum*.

Our general approach is to label all but a single 1-factor so that the partial sums in A are multiples of 3, while the partial sums in B are non-multiples of 3. At this stage no vertex of A conflicts with a vertex of B. We then label the final 1-factor with reserved labels that are multiples of 3 so that we resolve all potential conflicts within A and within B. Before we begin the general approach, we observe two facts that together show that 2-regular graphs are antimagic.

Fact 5.1. *[20] Every cycle is antimagic.*

Proof: Assign the labels to edges as $1, 3, \ldots, n, n-1, \ldots, 4, 2$ in order around an n-cycle (if n is odd; otherwise, n and $n-1$ are switched in the middle. The sums are $4, 8, \ldots, 10, 6, 3$; that is, the sums of consecutive odd integers are even multiples of 2, while the sums of consecutive even integers are odd multiples of 2. $\square$

Fact 5.2. *If G_1 and G_2 are each regular antimagic graphs, then the disjoint union of G_1 and G_2 is also antimagic.*

Proof: Index G_1 and G_2 so that vertices in G_2 have degree at least as large as those in G_1. Let $m_1 = |E(G_1)|$. Place an antimagic labeling on G_1, using the first m_1 labels. Label G_2 by adding m_1 to each label in an antimagic labeling of G_2.

Translating edge labels by m_1 adds $m_1 k$ to the sum at each vertex of G_2, so the new labeling of G_2 has distinct vertex sums. Hence there are no conflicts within G_1 and no conflicts within G_2. There are also no conflicts between a vertex in G_1 and one in G_2, since each vertex-sum in G_1 is less than $m_1 k$ and each vertex-sum in G_2 is greater than $m_1 k$. $\square$

More generally, given any labeling of a regular graph, adding the same amount to each label does not change the pairs of vertices that conflict. Fact 5.1 and Fact 5.2 immediately yield:

Corollary 5.3. *Every simple 2-regular graph is antimagic.*

We will consider odd and even degree separately. Although 2-regular graphs are easy, the general construction is a bit more complicated for even degree than for odd degree.

5.1 Regular bipartite graphs with odd degree

We have observed that a k-regular bipartite graph G decomposes into 1-factors. We can combine these 1-factors in any desired fashion. In particular, when k is odd and at least 5, we can decompose G into a $(2l+2)$-factor and a 3-factor, where $l \geq 0$. Our aim will be to combine special labelings of these two factors to obtain an antimagic labeling of G. The case $k = 3$ is handled separately; we do this before the general argument.

Theorem 5.4. *Every 3-regular bipartite graph is antimagic.*

Proof: Since G has $3n$ edges, we have the same number of labels in each congruence class modulo 3. For convenience, we use the term *j-labels* to designate the first n positive integers that are congruent to j modulo 3, where $j \in \{0, 1, 2\}$.

Decompose G into a 1-factor H_1 and a 2-factor H_2. We will reserve the 0-labels for H_1. We will label H_2 with the 1-labels and 2-labels so that the partial sum at each vertex of A is $3n$. We do this by pairing each 1-label i with the 2-label $3n - i$. These pairs have sum $3n$; at each vertex of A, we use the two integers in some pair. Subsequently, every assignment of 0-labels to H_1 yields distinct vertex-sums within A.

We have assigned a pair of labels at each vertex of A in H_2, but we have not decided which edge gets which label. Next we try to make this choice so that in H_2 the partial sums at vertices of B will not be multiples of 3. In each component of H_2, we will fail at most once.

Let C be a cycle that is a component of H_2. We have a 1-label and a 2-label at each vertex of A. As we follow C, if we have a 1-label and then a 2-label at a vertex of A, then the next vertex of A should have a 2-label followed by a 1-label (and vice versa), since the sum of two 1-labels or two 2-labels is not a multiple of 3. If $|V(C) \cap A|$ is even, then we succeed throughout; if $|V(C) \cap A|$ is odd, then at one vertex of C in B we will have a 1-label and a 2-label. Call such a vertex of B *bad*. A cycle in H_2 only has a bad vertex only if it has length at least 6, so at most $n/3$ vertices in B will be bad. Let m be the number of bad vertices.

To avoid conflicts between vertices of A and bad vertices of B, we will make the vertex-sum at each bad vertex smaller than at any vertex of A. Furthermore, we will make the partial sums in H_2 at these vertices equal. Consider the 1-labels and 2-labels from 1 through $3m - 1$; group them into pairs j and $3m - j$. The sum in each such pair is $3m$, which is at most n. Allocate the pairs for H_2 to vertices of A so that at each bad vertex of B, the labels are the small elements from pairs in the original pairing and form a pair with sum $3m$ in this most recent pairing.

Now we need to label H_1. We must achieve three goals: resolve all conflicts among the good vertices in B, resolve all conflicts among the bad vertices in B, and resolve all conflicts between A and the bad vertices in B.

We consider the last goal first. For every assignment of 0-labels to H_1, the vertex-sums in A will be $\{3n + 3, 3n + 6, \ldots, 6n - 3, 6n\}$. To ensure that the vertex-sums at the bad vertices in B will be less than $3n + 3$, we use the smallest 0-labels at the bad vertices. Since there are at most $n/3$ bad vertices, every 0-label at such a vertex is at most n. Thus, every sum at a bad vertex is at most $2n$, which is less than $3n$. Furthermore, the sums at bad vertices are $3m$ plus distinct 0-labels; hence they are distinct, which completes the second goal.

For the first goal, let $b_1, b_2, b_3, \ldots$ denote the good vertices of B in order of increasing partial sum from H_2 (there may be ties). We assign the remaining 0-labels to edges of H_1 at $b_1, b_2, \ldots$ in increasing order. Since the 0-labels are distinct, this prevents conflicts among the good vertices in B. $\qquad\square$

For larger even degree, we will construct an antimagic labeling from special labelings of two subgraphs. Like the labeling we constructed for 3-regular graphs, the first labeling will have equal sums at vertices of A, but this time we guarantee that all sums at vertices of B are not congruent modulo 3 to the sums at vertices of A.

Lemma 5.5. *If G is a $(2l+2)$-regular bipartite graph with parts A and B of size n, then G has a labeling such that the sum at each vertex of A is some fixed value t and the sum at each vertex of B is not congruent to t modulo 3.*

Proof: As remarked earlier, we can decompose G into a $2l$-factor H_{2l} and a 2-factor H_2. Let $m = (2l+2)n$; thus m is the largest label. Since m is even, we can partition the labels 1 through m into pairs that sum to $m+1$. With $m+1 \equiv 2a \pmod 3$, each pair consists of two elements in the same congruence class as a modulo 3 or elements in the two other congruence classes modulo 3. Call these *like-pairs* and *split-pairs*, respectively.

At each vertex of A, we will use l of these pairs as labels in H_{2l}. Thus each vertex of A will have partial sum $(m+1)l$ in H_{2l}; we will assign the pairs so that the partial sums in B are not congruent to $(m+1)l$ modulo 3. We use the pairs in which the smaller label ranges from 1 to ln. Note that H_{2l} decomposes into even cycles (for example, we can take $2l$ 1-factors two at a time to generate 2-factors whose union is H_{2l}).

For each cycle in the decomposition of H_{2l} into even cycles, at vertices of A we use pairs of labels of the same type: all like-pairs or all split-pairs. When using split-pairs, we assign the labels so that the same congruence class modulo 3 is always first. If we have all like-pairs or all split-pairs, this ensures that at each vertex of B, each cycle contributes an amount to the sum that is congruent to $2a$ modulo 3. There is at most one cycle where we are forced to use both like-pairs and split-pairs. Let x and y be the vertices of B where, in this cycle, we switch between like-pairs and split-pairs. At each vertex of A, the partial sum in H_{2l} is $(m+1)l$. At each vertex of B, except x and y, the partial sum is congruent to $(m+1)l$ modulo 3.

On H_2, we use the remaining pairs of labels so that we add $m+1$ to each partial sum in A, but what we add to each partial sum in B is not congruent to $m+1$ modulo 3. If we can do this (and treat x and y specially), then the sum at each vertex of A will be $(m+1)(l+1)$, while at each vertex of B the sum will be in a different congruence class modulo 3 from $(m+1)(l+1)$.

On each cycle, we use the pairs of labels that contain the smallest unused labels. Thus, every third pair we use is a like-pair; the others are split pairs. We begin with a like-pair and alternate using a like-pair and a split-pair until the like-pairs allotted to that cycle are exhausted. For the remaining split-pairs, we alternate them in the form $(a+1, a+2)$ followed by $(a+2, a+1)$; in this way the sum of the two labels used at any vertex of B is not congruent to $2a$ modulo 3. If no like-pair is available to be used on the cycle, then the cycle has length 4 and we label it with split-pairs in the form $(a+1, a+2), (a+2, a+1)$, and the same property holds.

One or two cycles in H_2 may contain the vertices x and y, where the sum in H_{2l} differs by 1 from a value congruent to $(m+1)l$ modulo 3. Suppose that the sums in H_{2l} at x and y are $(m+1)l+t_1$ and $(m+1)l+t_2$.

We want the sum at x in H_2 to be either $2a - t_1 + 1 \pmod 3$ or $2a - t_1 + 2$. Similarly, we want the sum at y in H_2 to be in $\{2a - t_2 + 1, 2a - t_2 + 2\}$. The more difficult case is when x and y lie on the same cycle in H_2. However, given the realization that we have two choices each for the sums (modulo 3) at x and y, it is not difficult to adapt the labeling given above for cycles of H_2 so that it applies in the current case as well.

At these vertices we want the contribution from H_2 to be congruent to $2a$ modulo 3. We deal with these first and can then make the argument above for the remaining cycles. If x and y lie on a single 4-cycle, then we use two like-pairs or two split-pairs ordered as $(a+1, a+2), (a+1, a+2)$. ****We must make sure that this does not leave an odd number of split-pairs for one ordinary cycle.*** If one or both of x and y lie on a longer cycle, then at each we put edges from two like-pairs or from two split-pairs ordered as $(a+1, a+2), (a+1, a+2)$. The remaining pairs, whether they are like-pairs or split-pairs as we allocate them to this cycle, can be filled in so that like-pairs are not consecutive anywhere else and neighboring split-pairs alternate their "orientation".

Thus the labeling of H_2 enables us to keep the overall sum at each vertex of B out of the congruence class of $(m+1)(l+1)$ modulo 3. □

Lemma 5.6. *If G is a 3-regular bipartite graph with parts A and B, where $B = \{b_1, \ldots, b_n\}$, then G has a labeling so that at each b_i the sum is $3n + 3i$, and for each i exactly one vertex in A has sum $3n + 3i$.*

Proof: Decompose G into three 1-factors: R, S, and T. In R, use label $3i - 2$ on the edge incident to b_i; let a_i be the other endpoint of this edge. In S, use label $3n + 3 - 3i$ on the edge incident to a_i; call the other endpoint of this edge b_i'. In T, use label $3i - 1$ on the edge incident to b_i'; call the other endpoint of this edge a_i'. Note that each 1-factor received the labels from one congruence class modulo 3.

The partial sum in $S \cup T$ at each vertex of B is $3n + 2$. Hence, the sum at b_i for all of G is $3n + 3i$. Similarly, the partial sum in $R \cup S$ at each vertex of A is $3n + 1$. Hence, the vertex-sum at a_i' is $3n + 3i$. □

Theorem 5.7. *Every regular bipartite graph of odd degree is antimagic.*

Proof: Let G be a regular bipartite graph of degree k. Theorem 5.4 is the case $k = 3$. For $k > 3$, let $k = 2l + 5$ with $l \geq 0$, and decompose the graph G into a $(2l+2)$-factor H' and a 3-factor H. Label H' as in Lemma 5.5; this uses labels 1 through $(2l+2)n$. Add $3n$ to each label, leaving labels 1 through $3n$ for H. Each vertex-sum increases by $9n$, which is a multiple of 3, so the congruence properties obtained in Lemma 5.5 remain true for the new labeling.

Let b_i denote the vertices of B in order of increasing partial sum in H'. Label H as in Lemma 5.6. Because all the partial sums in H are multiples of 3, the labeling of H' resolves each potential conflict between a vertex of A and a vertex of B. Because the b_i are in order of increasing partial sum in H', the labeling of H resolves all potential conflicts within B. Similarly, since the labeling of H' gives the same partial sum to all vertices of A, the labeling of H resolves all potential conflicts within A.

We have checked that the labeling is antimagic. □

5.2 Regular bipartite graphs with even degree

Lemma 5.8. *Let n be a positive integer. If n is even, then we can partition $\{1, 2, \ldots, 3n\}$ into triples such that the sum of each triple is $6n + 3$ or $3n$. If n is odd, then we can partition $\{1, 2, \ldots, 3n\}$ into triples such*

that the sum of each triple is 6n or 3n. Furthermore, each triple consists of one integer from each residue class modulo 3.

Proof: Suppose n is even. We partition the labels into triples such that the sum of each triple is either $3n$ or $6n + 3$. Consider the triples $(3n - 3i + 3, 3n - 3i + 2, 6i - 2)$ and $(3i, 3i - 1, 3n - 6i + 1)$ for $1 \leq i \leq n/2$. Triples of the first type sum to $6n + 3$ and triples of the second type sum to $3n$.

Suppose n is odd. We partition the labels into triples such that the sum of each triple is either $3n$ or $6n$. Consider the triples $(3n - 3i + 3, 3n - 3i + 2, 6i - 5)$ for $1 \leq i \leq \lceil n/2 \rceil$ and $(3i, 3i - 1, 3n - 6i + 1)$ for $1 \leq i \leq \lfloor n/2 \rfloor$. Triples of the first type sum to $6n$ and triples of the second type sum to $3n$. $\qquad\square$

Theorem 5.9. *Every regular bipartite graph of even degree at least 8 is antimagic.*

Proof: We decompose G into two 3-factors and a $(2l + 2)$-factor; call these G_3, H_3, and H_{2l+2}, respectively. We label H_{2l+2} as in Lemma 5.5, using all but the $6n$ smallest labels. This resolves every conflict between a vertex of A and a vertex of B.

We partition the labels $\{3n + 1, 3n + 2, \ldots, 6n\}$ into triples as in Lemma 5.8. In G_3, at each vertex of A we will use the the three labels of some triple. To ensure the sum at each vertex of B is $0(\bmod\ 3)$, we do the following. Partition the 3-factor into three 1-factors; We use 0-labels on the first 1-factor, $1(\bmod\ 3)$ labels on the second 1-factor, and $2(\bmod\ 3)$ labels on the third 1-factor.

Now consider the partial sums in the union of H_{2l+2} and G_3; let b_i denote the vertices of B in order of increasing partial sum. Label H_3 as in Lemma 5.6. This resolves every conflict between two vertices in the same part. Hence, the labeling is antimagic. $\qquad\square$

Lemma 5.10 is very similar to Lemma 5.8. Lemma 5.10 serves the same role in the proof of Theorem 5.9 that Lemma 5.8 does in the proof of Theorem 5.11.

Lemma 5.10. *Let n be a positive integer. Let H be the set of positive labels less than $4n$ that are not 0 modulo 4, i.e. $H = \{1, 2, 3, 5, 6, \ldots, 4n - 2, 4n - 1\}$. If n is even, then we can partition H into triples such that the sum of each triple is either $4n - 2$ or $8n + 2$. If n is odd, then we can partition H into triples such that the sum of each triple is either $4n - 2$ or $8n - 2$. Furthermore, each triple consists of integer from each nonzero residue class modulo 4.*

Proof: Suppose n is even. We have triples of the form $(8i - 3, 4n - 4i + 2, 4n - 4i + 3)$, with $1 \leq i \leq n/2$, and triples of the form $(4n - 8i + 1, 4i - 6, 4i - 5)$, with $1 \leq i \leq n/2$. It is easy to see that triples of the first form sum to $8n + 2$ and that triples of the second form sum to $4n - 2$. It is straightforwad to verify that these triples partition H.

Suppose n is odd. We have triples of the form $(8i - 7, 4n - 4i + 2, 4n - 4i + 3)$, with $1 \leq i \leq \lceil n/2 \rceil$, and triples of the form $(4n - 8i + 5, 4i - 2, 4i - 1)$, with $1 \leq i \leq \lfloor n/2 \rfloor$. It is easy to see that triples of the first form sum to $8n - 2$ and that triples of the second form sum to $4n - 2$. It is straightforwad to verify that these triples partition H. $\qquad\square$

Theorem 5.11. *Every 6-regular bipartite graph is antimagic.*

Proof: Throughout this proof we assume n is odd. The argument is analagous when n is even, so we omit the details. We decompose G into a 3-factor, a 2-factor, and a 1-factor. We label the 3-factor with the labels that are less than $4n$ and are not 0 modulo 4, so that the partial sum at each vertex of B is $2(\bmod\ 4)$ and the partial sum at each vertex of A is $4n-2$ or $8n-2$. To do this we partition the labels for the 3-factor into triples as specified in Lemma 5.10.

At each vertex of A, we use the three labels in some triple. More exactly, we decompose the 3-factor into three 1-factors; we use $1(\bmod\ 4)$ labels on the first 1-factor, use $2(\bmod\ 4)$ labels on the second 1-factor, and use $3(\bmod\ 4)$ labels on the third 1-factor. This ensures that the partial sum at each vertex of B is $2(\bmod\ 4)$.

We label the 2-factor with the labels $4n+1$ through $6n$, so that the partial sum at each vertex of A is $10n+1$ and the sum at each vertex of B is $\not\equiv 10n+1(\bmod\ 4)$. To do this, we partition the labels for the 2-factor into pairs that sum to $10n+1$. We consider the labels in each pair modulo 4. We have two types of pairs: $(1,2)$ pairs and $(3,0)$ pairs (since n is odd).

We want to avoid using two labels at a vertex of B that sum to $3(\bmod\ 4)$. We choose the pairs of labels to use on each cycle arbitrarily, except that each cycle must use at least one $(1,2)$ pair and at least one $(3,0)$. We first use all the $(1,2)$ pairs, alternating them as $(1,2),(2,1),(1,2),(2,1),\ldots$, then use all the $(3,0)$ pairs, alternating them as $(3,0),(0,3),(3,0),(0,3),\ldots$. As long as we use at least one $(1,2)$ pair and one $(3,0)$ pair on each cycle of the 2-factor, we have no problems. Since we use at least one $(1,2)$ pair and one $(3,0)$ pair on each cycle of the 2-factor, we are able to avoid vertex sums in B that are congruent to $3(\bmod\ 4)$.

Now we consider partial sums in the 5-factor that is already labeled. The partial sum at each vertex of A is $4n-2$ or $8n-2$. The partial sum at each vertex of B is not congruent to 2 modulo 4. The labels we will use on the final 1-factor are all multiples of 4. So, regardless of how we label the final 1-factor, no vertex in A will conflict with any vertex in B. We call a vertex in A *small* if it's partial sum in the 5-factor is $4n-2$; otherwise, we call it *big*. It is clear that regardless of how we label the final 1-factor, no big vertex will conflict with another big vertex; similarly, no small vertex will conflict with a small vertex. Observe that the largest possible sum at a small vertex is $4n-2+4n=8n-2$. The smallest possible sum at a big vertex is $8n-2+4=8n+2$. Hence, no small vertex will conflict with a big vertex. Thus, we choose the labels for the final 1-factor to ensure that no two vertices in B conflict.

Let b_i denote the vertices of B in order of increasing partial sum in the 5-factor. In the final 1-factor, we use label $4i$ at vertex i. This ensures that vertex-sums in B are distinct. Thus, the labeling is antimagic. $\qquad\square$

The proof for 4-regular graphs is more complicated than for 6-regular graphs. In the 6-regular case, we labeled the 2-factor to ensure there were no conflicts between any vertex in A and any vertex in B; we labeled the 1-factor and the 3-factor to ensure there were no conflicts between two vertices in the same part. The proof for 4-regular graphs is similar, but since we have one less 2-factor, we cannot ensure that all vertex-sums in B differ modulo 4 from the vertex-sums in A. So similar to the 3-regular graphs, we introduce *good* and *bad* vertices in B. We handle bad vertices in a similar way to the case of the 3-regular graphs.

Theorem 5.12. *Every 4-regular bipartite graph is antimagic.*

Proof: Throughout this proof we assume n is odd. The argument is analagous when n is even, so we omit the details. We decompose G into a 3-factor and a 1-factor. We label the 3-factor with the 1-labels, 2-labels,

and 3-labels that are less than $4n$, so that the partial sum at each vertex of B is $4n-2$ or $8n-2$. To do this, we partition the labels for the 3-factor into triples as specified in Lemma 5.10. At each vertex in A, we will use the three labels of a triple. Consider a vertex of B: if its partial sum in the 2-factor is $2(\bmod 4)$, then we call the vertex *bad*; otherwise, we call it *good*. We assign the labels of each triple to the edges at a vertex of A to minimize the number of bad vertices in B. Initially, we only assign to each edge an equivalence class: $1(\bmod 4)$, $2(\bmod 4)$, or $3(\bmod 4)$. This determines which vertices in B are bad. We will then assign the labels to edges to minimize the largest partial sum at a bad vertex of B. Since the bad vertices in B will have vertex-sums in the same equivalence class (modulo 4) as the vertex-sums in A, to avoid conflicts we will ensure that the vertex-sum at every bad vertex is smaller than the smallest vertex-sum in B.

We begin by decomposing the 3-factor into three 1-factors. We label each edge in the first 1-factor with a 1, each edge in the second 1-factor with a 2, and each edge in the third 1-factor with a 3. However, this makes every vertex in B bad. To fix this, we consider the 2-factor labeled with 1s and 2s; specifically consider a single cycle in this 2-factor. Select a vertex of A on the cycle, then select every second vertex of A along the cycle; at each of the selected vertices, swap the labels 1 and 2 on the incident edges. If the cycle has length divisible by 4, then all of its vertices are now good. If the length is not divisible by 4, then one bad vertex will remain. Note that a cycle has a bad vertex only if its length is at least 6. So, at most $n/3$ vertices are bad. We now reduce the number of bad vertices further, as follows.

If a vertex is bad, consider the incident edge labeled 3, and the edge labeled 2 that is adjacent in A to this first edge; these two edges form a *bad path*. We will swap the two labels on a bad path to reduce the number of bad vertices. Consider the graph induced by bad paths; each component is a path or a cycle. In a path component, we swap the labels on every second bad path; this fixes all the bad vertices. We handle cycle components similarly, although in each cycle one bad vertex may remain (similar to the previous step). Thus, after this step, at most $1/3$ of the previously bad vertices remain bad. So, at most $n/9$ vertices remain bad. We also need to verify that when we swap the labels on a bad path, no good vertex becomes bad.

If a good vertex has partial sum $3(\bmod 4)$, we call it *heavy*; if it has partial sum $1(\bmod 4)$, we call it *light*. Before we swap the labels on any bad path, the triple of labels incident to a light vertex is $(1,1,3)$; the triple incident to a heavy vertex is $(2,2,3)$. Thus, we do not swap any labels incident to a light vertex. However, the labels incident to a heavy vertex could become $(2,3,3)$ or even $(3,3,3)$. In each case though, the vertex remains good.

Finally, if any vertex in A is adjacent to two or more bad vertices, we swap the labels on its incident edges to make each vertex good. Thus, we have at most $n/9$ bad vertices and each vertex in A is adjacent to at most one bad vertex. Now we assign the actual labels to the edges (rather than only the equivalence classes) so that the partial sum at each bad vertex is small. We assign the $n/9$ smallest $1(\bmod 4)$ labels to be incident to the bad vertices; the largest is less than $4n/9$. Similarly, we assign the $n/9$ smallest $2(\bmod 4)$ labels to be incident to the bad vertices; again the largest is less than $4n/9$. Each time we assign a label, we also assign the other labels in its triple. Since each $2(\bmod 4)$ label is in a triple with the $3(\bmod 4)$ label one greater, the $n/9$ smallest $3(\bmod 4)$ labels are already assigned. So we assign the next $n/9$ smallest $3(\bmod 4)$ labels to be incident to the bad verties; the largest of these labels is less than $8n/9$. Finally, we will assign the $n/9$ smallest $0(\bmod 4)$ labels to be incident to the bad vertices. Thus, the largest vertex-sum at a bad vertex is less than $3(4n/9)+8n/9 < 3n$. Hence, no bad vertex will conflict with any vertex in A.

To ensure that no two bad vertices conflict, we assign the labels to the final 1-factor in order of increasing partial sum at the bad vertices. After we assign all the labels incident to the bad vertices, we assign the remaining labels incident to the good vertices, again in order of increasing partial sum in B. This ensures that no two good vertices conflict. If the partial sum at a vertex of A is $4n - 2$ we call it *small*; otherwise we call it *big*. After we assign the labels on the final 1-factor, the smallest possible vertex-sum at a big vertex is $(8n - 2) + 4 = 8n + 2$; the largest possible sum at a small vertex is $(4n - 2) + 4n = 8n - 2$. So no small vertex conflicts with a big vertex. Additionally, all the small vertex-sums are distinct; so are the large vertex-sums. Thus, the labeling is antimagic. $\qquad\square$

Bibliography

[1] N. Alon, G. Kaplan, A. Lev, Y. Roditty, and R. Yuster, *Dense graphs are antimagic*, J. Graph Theory, **47** (2004) pp. 297–309.

[2] L.D. Andersen, *The strong chromatic index of a cubic graph is at most 10*, Topological, algebraical and combinatorial structures. Frolík's memorial volume. Discrete Math., **108** (1992), 231–252.

[3] A. Berliner, U. Bostelmann, R.A. Brualdi, and L. Deatt, *Sum list colorings graphs*, Graphs Combin., **22** (2006), 173–183.

[4] B. Bollabás and A.J. Harris, *List-colorings of graphs*. Graphs Combin., **1** (1985), 115–127.

[5] O.V. Borodin, A.V. Kostochka, and D.R. Woodall, *List edge and list total colourings of multigraphs*, J. Combin. Theory Ser. B, **71** (1997), 184–204.

[6] O.V. Borodin, *Generalization of a theorem of Kotzig and a prescribed coloring of the edges of planar graphs*, Mathematical Notes of the Academy of Sciences of the USSR, **48** (1990), 1186–1190.

[7] O.V. Borodin, A.N. Glebov, A.O. Ivanova, T.K. Neustroeva, and V.A. Tashkinov, *Sufficient conditions for planar graphs to be 2-distance $(\Delta + 1)$-colorable. (Russian)* Sib. Elektron. Mat. Izv., **1** (2004), 129–141.

[8] W.G. Brown, *On graphs that do not contain a Thomsen graph*, Canad. Math. Bull., **9** (1966), 281–285.

[9] A. Cavicchioli, M. Meschiari, B. Ruini, and F. Spaggiari, *A Survey on Snarks and New Results: Products, Reducibility and a Computer Search*, J. Graph Theory, **28** (1998), 57–86.

[10] F.R.K. Chung, A. Gyárfás, Zs. Tuza, and W.T. Trotter, *The maximum number of edges in $2K_2$-free graphs of bounded degree*, Discrete Math., **81** (1990), 129–135.

[11] Z. Dvořák, R. Škrekovski, and M. Tancer, *List-coloring squares of sparse subcubic graphs*, submitted for publication.

[12] M. N. Ellingham and L. Goddyn, *List edge colourings of some 1-factorable multigraphs*, Combinatorica, **16** (1996), 343–352.

[13] P. Erdös, A.L. Rubin, and H. Taylor, *Choosability in graphs*, Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing, Arcata, California, 1979, Congressus Numeratium, **26** (1980), 125–157.

[14] P. Erdös, S. Fajtlowicz, and A.J. Hoffman, *Maximum degree in graphs of diameter 2*, Networks, **10** (1980), 87–90.

[15] R.J. Faudree, A. Gyárfás, R.H. Schelp, and Zs. Tuza, *The strong chromatic index of graphs*, Ars Combinatoria, **29B** (1990), 205–211.

[16] H. Fleischner and M. Stiebitz, *A solution to a colouring problem of P. Erdös.* Special volume to mark the centennial of Julius Petersen's "Die Theorie der regulären Graphs", Part II. Discrete Math., **101** (1992), 39–48.

[17] J. Gallian, *A Dynamic Survey of Graph Labeling*, Elec. J. Combinatorics, `www.combinatorics.org/Surveys/ds6.pdf`, (2007).

[18] F. Galvin, *The list chromatic index of a bipartite multigraph*, J. Combin. Theory Ser. B, **63** (1995), 153–158. k

[19] R.P. Gupta, *The chromatic index and the degree of a graph (Abstract 66T-429)*, Notices Amer. Math. Soc., **13** (1966) 719.

[20] N. Hartsfield and G. Ringel, *Pearls in Graph Theory: A Comprehensive Introduction.* Academic Press, 1990.

[21] F. Havet, *Choosability of the square of planar subcubic graphs with large girth*, Technical Report, INRIA, 2006.

[22] R. Häggkvist and J. Janssen, *New bounds on the list chromatic index of the complete graph and other simple graphs*, Combin. Probab. Comput., **6** (1997), 295–313.

[23] A.J. Harris, *Problems and conjectures in extremal graph theory*, Ph.D. Dissertation, Cambridge University, UK, 1984.

[24] P. Haxell and R. Naserasr, *Edge-choosability of the Petersen Graph*, preprint.

[25] B. Heinold, *Sum List Coloring*, Ph.D. Dissertation, Lehigh University, 2006.

[26] A.J. Hoffman and R.R. Singleton, *On Moore graphs with diameters 2 and 3*, IBM Journal of Research and Development, **4(5)** November 1960, 497–504.

[27] P. Horák, *The strong chromatic index of graphs with maximum degree four*, Contemporary methods in graph theory, (1990), 399–403, Bibliographisches Inst., Mannheim.

[28] M.N. Huxley, *On the difference between consecutive primes*, Invent. Math. **15** (1972), 164–170.

[29] G. Isaak, *Sum list coloring $2 \times n$ arrays*, Elec. J. Combinatorics, **9** (2002), Note 8.

[30] G. Isaak, *Sum list coloring block graphs*, Graphs Combin., **20** (2004), 499–506.

[31] T.R. Jensen and B. Toft, *Graph Coloring Problems*, John Wiley & Sons, Inc., 1995.

[32] M. Juvan, B. Mohar, and R. Škrekovski, *List total colourings of graphs*, Combina. Probab. Comput., **7** (1998), 181–188.

[33] M. Juvan, B. Mohar and R. Škrekovski, *Graphs of degree 4 are 5-edge-choosable*, J. Graph Theory, **32** (1999), 250–264.

[34] A.V. Kostochka, *List edge chromatic number of graphs with large girth*, Discrete Math. **101** (1992), 189–201.

[35] A.V. Kostochka and D.R. Woodall, *Choosability conjectures and multicircuits.* Discrete Math., **240** (2001), 123–143.

[36] Z. Dvořák, D. Král, P. Nejedlý, and R. Škrekovski, *Coloring squares of planar graphs with no short cycles*, submitted for publication.

[37] M. Miller and J. Širáň, *Moore graphs and beyond: A survey of the degree/diameter problem*, `www.combinatorics.org/Surveys/ds14.pdf` (2005), 12.

[38] B. Mohar, (r,s)-*edge-choosability of cubic graphs*, `http://www.fmf.uni-lj.si/~mohar/Problems.html`, June 2003.

[39] M. Molloy and M. Salavatipour, *A bound on the chromatic number of the square of a planar graph*, J. Combin. Theory Ser B., **94** (2005), 189–213.

[40] S. Skulrattanakulchai and H.N. Gabow, *Coloring Algorithms on Subcubic Graphs*, Internat. J. Found. Comput. Sci., **15** (2004), 21–40.

[41] C. Thomassen, The square of a planar cubic graph is 7-colorable, *manuscript*.

[42] Zs. Tuza and M. Voigt, *Every 2-choosable graph is (2m,m)-choosable*, J. Graph Theory, **22** (1996), 245–252.

[43] Zs. Tuza and M. Voigt, *On a conjecture of Erdös, Rubin, and Taylor*, Tatra Mountains Math. Publ., **9** (1996), 69–82.

[44] V.G. Vizing, *On an estimate of the chromatic class of a p-graph*, Diskret. Analiz., **3** (1964), 25–30.

[45] V.G. Vizing, *Critical graphs with a given chromatic class*, Diskret. Analiz., **5** (1965), 9–17.

[46] W. Wang and K. Lih, *Choosability and edge choosability of planar graphs without five cycles*, Appl. Mathematics Letters, **15** (2002), 561–565.

[47] W. Wang and K. Lih, *Choosability and edge choosability of planar graphs without intersecting triangles*, SIAM J. Discrete Math., **15** (2002), 538–545.

[48] W. Wang and K. Lih, *Light subgraphs and edge choosability of planar graphs without 3-cycles or 4-cycles*, Tamsui Oxford J. Math. Sci., **20** (2004), 353–376.

[49] W. Wang and K. Lih, *Labeling planar graphs with conditions on girth and distance two*, SIAM J. Discrete Math., **17** (2003), 264–275.

[50] G. Wegner, *Graphs with given diameter and a coloring problem, preprint*, University of Dortmund, Dortmund, Germany, 1977.

[51] P. Wernicke, *Über den kartographischen Vierfarbensatz (German)*, Math. Ann., **58** (1904), 413–426.

[52] D.B. West, *Introduction to Graph Theory*. Prentice Hall, second edition, 2001.

[53] D. B. West, *Strong Edge-Coloring*, Open Problems - Graph Theory and Combinatorics. `http://www.math.uiuc.edu/~west/openp/strongedge.html`, 2003.

[54] D.R. Woodall, *List colourings of graphs*, Surveys in Combinatorics, J.W.P. Hirschfeld ed., London Math. Soc. Lecture Notes Series **288**, Cambridge University Press, (2001) 269–301.

[55] L. Zhang and B. Wu, *Edge choosability of planar graphs without small cycles*, Discrete Math., **283** (2004), 289–293.

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