Subcubic edge chromatic critical graphs have many edges

Daniel W. Cranston*  Landon Rabern†

June 13, 2015

Abstract

We consider graphs $G$ with $\Delta = 3$ such that $\chi'(G) = 4$ and $\chi'(G - e) = 3$ for every edge $e$, so-called critical graphs. Jakobsen noted that the Petersen graph with a vertex deleted, $P^*$, is such a graph and has average degree only $2 + \frac{2}{3}$. He showed that every critical graph has average degree at least $2 + \frac{2}{3}$, and asked if $P^*$ is the only graph where equality holds. We answer his question affirmatively. Our main result is that every subcubic critical graph, other than $P^*$, has average degree at least $2 + \frac{26}{37} = 2.702$.

1 Introduction

A proper edge-coloring of a graph $G$ assigns a color to each edge in $E(G)$ such that edges with a common endpoint receive distinct colors. The minimum number of colors needed for a proper edge-coloring is the edge-chromatic number of $G$, denoted $\chi'(G)$. The maximum degree of $G$ is denoted $\Delta(G)$, or simply $\Delta$ when $G$ is clear from context. Note that always $\chi'(G) \geq \Delta(G)$. Vizing famously proved that $\Delta(G) + 1 \geq \chi'(G) \geq \Delta(G)$ for every graph $G$. A graph is edge-chromatic critical (also $\Delta$-critical, or simply critical) if $\chi'(G) > \Delta(G)$ but $\chi'(G - e) = \Delta(G)$ for every edge $e$. A vertex of degree $k$ is a $k$-vertex. If $v_1$ is a $k$-vertex and $v_1$ is adjacent to $v_2$, then $v_1$ is a $k$-neighbor of $v_2$.

Vizing [8, 9] was the first to seek a lower bound on the number of edges in a critical graph, in terms of its number of vertices. This problem is now widely studied, for a large range of maximum degrees $\Delta$. Woodall gives a nice history of this work, in the introduction to [10]. In this paper, we study the problem for subcubic graphs, i.e., when $\Delta = 3$.

It is easy to check that 2-critical graphs are precisely odd cycles, which are completely understood. So the first non-trivial case is 3-critical graphs. Let $P^*$ denote the Petersen graph with a vertex deleted. Jakobsen [4] observed that $P^*$ is 3-critical and has average degree $2 + \frac{2}{3}$. In the same paper, he asked if every 3-critical graph has average degree at least $2 + \frac{2}{3}$. A year later [5], he answered this question affirmatively. However, in this second paper Jakobsen asked whether this bound holds with equality for any graph other than $P^*$.

---

*Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA; dcranston@vcu.edu; Research of the first author is partially supported by NSA Grant 98230-15-1-0013.

†LBD Data Solutions, Lancaster, PA; landon.rabern@gmail.com
A natural extension of this question is to determine the minimum $\alpha$ such that there exists an infinite sequence of 3-critical graphs with average degree at most $2 + \alpha$. The first progress toward answering this question is due to Fiorini and Wilson [3, p. 43], who constructed an infinite family of 3-critical graphs with average degree approaching $2 + \frac{3}{4}$ from below. Woodall [10, p. 815] gave another family with the same number of edges and vertices; see Figure 1. Before presenting his construction, we need a definition.

**Lemma 1.** If $G_1$ and $G_2$ are $k$-critical graphs, and $G$ is a Hajós join of $G_1$ and $G_2$ that has maximum degree $k$, then $G$ is also $k$-critical.

This is an old result of Jakobsen [4]. It is a straightforward exercise, so we omit the details, which are available in Fiorini & Wilson [2, p. 82–83] and Stiebitz et al. [7, p. 94].

**Corollary 2.** Let $G_1$ and $G_2$ be subcubic graphs, and let $G_1$ be 3-critical. If $G$ is a subcubic graph that is a Hajós join of $G_1$ and $G_2$, then $G$ is 3-critical if and only if $G_2$ is 3-critical.

**Proof.** The “if” direction follows immediately from the previous lemma. To prove the “only if” direction, we can assume that $\chi'(G_2) = 3$ and construct a 3-coloring of $G$ from 3-colorings of $G_2$ and $G_1 - v_1v_2$. \qed

Now we present Woodall’s construction of 3-critical graphs.

**Example 1.** Form $J_k$ by starting with $P^*$ and taking the Hajós join with $P^*$ a total of $k$ times (successively), so that each intermediate graph has $\Delta = 3$. The resulting graph $J_k$ is 3-critical, has $11k + 12$ edges and $8k + 9$ vertices. Thus, the average degree of the sequence $J_k$ approaches $2 + \frac{3}{4}$ from below.

The vertex and edge counts follow immediately by induction. That $J_k$ is 3-critical uses induction and also Lemma 1. \qed

Our main result is that every 3-critical graph $G$, other than $P^*$, has average degree at least $2 + \frac{26}{37}$. Before we prove this, it is helpful to provide a brief outline. Our proof uses the discharging method. More precisely, we first show that numerous subgraphs, not necessarily
induced, are forbidden from appearing in a minimal counterexample $G$. To conclude, we give each vertex a charge equal to its degree. Under the assumption that $G$ contains none of the forbidden subgraphs, we redistribute charge, without changing its sum, so that each vertex has final charge at least $2 + \frac{26}{37}$. This proves the theorem.

The intuition behind our proof is to show that every vertex in the graph has many “nearby” 3-vertices and not too many nearby 2-vertices. Let $H$ be the subgraph of $G$ induced by 3-vertices with 2-neighbors. To facilitate the discharging, we show that each component of $H$ is small. Further, for each 3-vertex $v$ not in $H$, we show that the sum of the sizes of the adjacent components of $H$ is small, so $v$ can give much of its extra charge to each vertex in these components of $H$.

We mentioned above that our proof begins by forbidding certain subgraphs from appearing in a critical graph. The easiest example of this is that no critical graph contains a 1-vertex. If so, we delete its incident edge $e$, color $G - e$ by criticality, then extend the coloring to $e$. More standard examples often require that we recolor part of the graph before we extend the coloring. An $(x, y)$-Kempe chain is a component of the subgraph induced by the edges colored $x$ and $y$. Note that each Kempe chain is either a path or an even cycle. If vertices $v_1$ and $v_2$ lie in the same $(x, y)$-Kempe chain, then $v_1$ and $v_2$ are $x, y$-linked. Given a proper coloring of (some subgraph of) a graph $G$, if we interchange the colors on some $(x, y)$-Kempe chain, the resulting coloring is again proper. This interchange is an $(x, y)$-Kempe swap and plays a central role in most proofs of forbidden subgraphs in critical graphs. If a color $w$ is used on an edge incident to a vertex $v$, then we say that $v$ sees $w$; otherwise $v$ misses $w$.

Suppose that $d(v_1) = 2$, $d(v_2) = 3$, and $v_1v_2 \in E(G)$. Suppose also that we 3-color $G - v_1v_2$ with colors $x$, $y$, and $z$. If we cannot extend this coloring to $G$, then (by symmetry) we may assume that $v_1$ sees $x$ and that $v_2$ sees $y$ and $z$. Furthermore, $v_1$ and $v_2$ must be $x, y$-linked; otherwise we perform an $(x, y)$-Kempe swap at $v_1$ and afterwards color $v_1v_2$ with $x$. Similarly, $v_1$ and $v_2$ must be $x, z$-linked. The quintessential tool for forbidding a subgraph in a critical graph is Vizing’s Adjacency Lemma, which he proved using Kempe chains and a similar structure for recoloring, known as Vizing fans.

**Vizing’s Adjacency Lemma.** Let $G$ be a $\Delta$-critical graph. If $u, v \in V(G)$ and $uv \in E(G)$, then the number of $\Delta$-neighbors of $u$ is at least $\max\{2, \Delta - d(v) + 1\}$.

The proof is available in Fiorini & Wilson [2, p. 72–74] and in Stiebitz et al. [7]. In the case $\Delta = 3$, we conclude that every 3-vertex has at most one 2-neighbor. This is helpful in our goal to prove that every 3-vertex has many nearby 3-vertices.

Two of our proofs that certain subgraphs are forbidden from $G$ are a bit lengthy. To improve readability, we simply state the results when we need them in Section 2 (as Claims 2 and 5), and defer the proofs to Section 3. By using a computer, we were able to improve our edge bound for 3-critical graphs to $|E(G)| \geq (2 + \frac{42}{59})|V(G)|$. However, a human-readable proof is too long to include here (roughly 100 pages). We discuss this work a bit more in the Section 4 as well as give a web link where that proof is available.
2 Proof of Main Theorem

Main Theorem. Let $P^*$ denote the Petersen graph with a vertex deleted. If a graph $G$ with \( \Delta = 3 \) is critical, then either \( 2|E(G)| \geq (2 + \frac{26}{37})|V(G)| \) or else either $G$ is $P^*$.

Proof. We will prove the following variation, from which the version stated above follows: If $G$ is 3-critical then either \( 2|E(G)| \geq (2 + \frac{26}{37})|V(G)| \) or else $G = P^*$ or $G$ is the Hajós join of $P^*$ and a smaller 3-critical graph. To see that this implies the version stated above, consider a smallest 3-critical graph $G$. Since it is smallest, either $G$ has average degree at least $2 + \frac{26}{37}$ or else $G$ is $P^*$. Note that if $G$ has average degree less than $2 + \frac{3}{4}$, then the Hajós join of $G$ and $P^*$ has average degree higher than that of $G$ (since it has 6 more 3-vertices and two more 2-vertices). Thus, every 3-critical graph with average degree less than $2 + \frac{26}{37}$ must be formed from $P^*$ by repeatedly taking the Hajós join with further copies of $P^*$; these are Woodall’s construction, from Example[1]. It is easy to check that already $J_1$ has average greater than $2 + \frac{26}{37}$. Now we prove this variation of the theorem.

Suppose the theorem is false, and let $G$ be a minimal counterexample. Note, as follows, that $G$ is 2-edge-connected, so has minimum degree 2. If $G$ is disconnected, then we can color each component by minimality. Similarly, suppose $G$ has a cut-edge $v_1v_2$. By minimality, we can color $G - v_1v_2$, and permute the colors so that the same color is missing from $v_1$ and $v_2$. Before giving the discharging, we prove some structural claims about $G$ and $H$.

Claim 1. $G$ has no adjacent 2-vertices, no 3-vertex with two or more 2-neighbors, and no 3-cycle.

If $G$ has adjacent 2-vertices $v_1$ and $v_2$, then color $G - v_1v_2$. Now at most two colors are forbidden on $v_1v_2$, so we can extend the coloring. Recall that Vizing’s Adjacency Lemma guarantees that each 3-vertex has at least two 3-neighbors, so at most one 2-neighbor. Now suppose that $G$ has a 3-cycle $v_1v_2v_3$. First, suppose that $v_1v_2$ lies in a second 3-cycle $v_1v_2v_4$. If $v_3v_4$ is also in $G$, then $G \cong K_4$, so $\chi'(G) = 3$. So suppose not. Let $v_5$ be a neighbor of $v_3$ other than $v_1$ and $v_2$ (or a neighbor of $v_4$ other than $v_1$ and $v_2$). To form $G'$ from $G$, contract $\{v_1, v_2, v_3, v_4, v_5\}$ to a single vertex. Since $G$ has no cut-edges, each of vertices $v_1, v_2, v_3, v_4$ has degree 3; thus, the average degree of $G'$ is less than that of $G$. Now by minimality, $\chi'(G') = 3$, and we can extend this coloring of $G'$ to $G$. So we may assume that no edge of $v_1v_2v_3$ lies on a second triangle. To form $G'$ from $G$, contract the three edges of triangle $v_1v_2v_3$. Again, by minimality, $\chi'(G') = 3$, and we can extend the coloring to $G$. □

Figure 2: Three subgraphs forbidden from a 3-critical graph $G$. Vertices drawn as rectangles have degree 2 in $G$ and those drawn as circles have degree 3 in $G$. 

4
Claim 2. The subgraphs shown in Figures 2(a), 2(b), and 2(c) are forbidden.

The proof for Figure 2(a) is given in Lemma 4 in Section 3. For Figures 2(b) and 2(c), we give the proof here. We begin with Figure 2(b). By criticality, we use colors $x$, $y$, and $z$ to color $G - v_3v_5$; call the coloring $\varphi$. WLOG, $v_5$ sees $x$, $\varphi(v_2v_3) = y$, and $\varphi(v_3v_4) = z$. If $v_1$ misses $x$, then $\varphi(v_1v_2) = z$ and $\varphi(v_1v_4) = y$. Now do an $(x,y)$-Kempe swap at $v_5$. Edge $v_3v_5$ will be colorable unless the $(x,y)$-Kempe path starting at $v_5$ ends at $v_3$, so assume that it does. Now $v_5$ sees $y$ and $v_1$ sees $y$. Thus, in the orginal case, we can assume that $v_1$ sees $x$. WLOG, $\varphi(v_1v_2) = x$. Uncolor the edges incident to $v_1$ and $v_3$ and color $v_2v_3$ with $x$. Now greedily color $v_2v_1, v_1v_4, v_1v_3$, and $v_3v_5$.

Now consider Figure 2(c). As above, we use colors $x$, $y$, $z$ to 3-color $G - v_3v_6$; call the coloring $\varphi$. WLOG, $v_6$ sees $x$. Since $\varphi(v_1v_2) \neq \varphi(v_1v_4)$, by symmetry, assume $\varphi(v_2v_5) = x$. Again, WLOG, $\varphi(v_2v_3) = y$, $\varphi(v_3v_4) = z$, and $\varphi(v_1v_2) = z$. We may assume that $v_3$ and $v_6$ are $x,y$-linked. Thus, $v_5$ sees $x$. If $\varphi(v_1v_4) = y$, then do a $(y,z)$-Kempe swap at $v_3$ (the entire component is just the 4-cycle $v_1v_2v_3v_4$). Now $v_3$ and $v_6$ are no longer $x,z$-linked, so do an $(x,z)$-Kempe swap at $v_3$, and color $v_3v_6$ with $z$. Thus, we assume that $\varphi(v_1v_4) = x$. Now again, do an $(x,z)$-Kempe swap at $v_3$, then color $v_3v_6$ with $z$. \qed

Recall that $H$ is the subgraph of $G$ induced by 3-vertices with 2-neighbors. A $t$-component of $H$ is a component of order $t$.

Claim 3. Each component of $H$ is a path on at most 5 vertices.

Suppose not. By construction, $\Delta(H) \leq 2$; since $G$ has no 3-cycle, assume that some component $H_1$ of $H$ induces a path or cycle $v_1 \ldots v_k$ on 4 or more vertices. We consider a path first; the case of a cycle is similar and easier. Since Figure 2(a) is forbidden, every set of four successive 3-vertices on the path must contain a pair with a common 2-neighbor. Since $G$ has no 3-cycles, no successive 3-vertices on the path have a common 2-neighbor. Similarly, since Figure 2(b) is forbidden, no vertices at distance two on the path have a common 2-neighbor. Thus, each path vertex $v_i$ (for $i \in \{1, \ldots, t - 3\}$) must share a common 2-neighbor with $v_{i+3}$; otherwise, we get the configuration in Figure 2(a) or Figure 2(b). This immediately gives that $t \leq 6$, since otherwise $v_4$ must share a common 2-neighbor with both $v_1$ and $v_7$, a contradiction. If $H_1$ is a path on 6 vertices, then $G$ is the Hajós join of $P^*$ and a smaller graph $J$. Since $G$ is 3-critical, Corollary 2 implies that $J$ is also 3-critical. This contradicts our hypothesis. Thus, $H_1$ cannot be a path on 6 vertices.

To rule out a cycle, note that we can’t pair up the 3-vertices so that $v_i$ and $v_{i+3}$ have a common 2-neighbor for each of the paths obtained by deleting a single cycle edge (since Figure 2(a) and Figure 2(b) need not be induced). If $H_1$ is a 4-cycle or 5-cycle, then any pairing gives a triangle or Figure 2(b); no pairing gives Figure 2(a). If $H_1$ is a 6-cycle, then we have only one possible pairing, but now the whole graph is $P^*$, which contradicts our hypothesis. Thus, $H_1$ must be a path on 5 or fewer vertices. \qed

Claim 4. No 3-vertex has two neighbors in the same component of $H$.

Suppose that $G$ contains such a 3-vertex $v$, and let $H_1$ be the component of $H$ containing two of its neighbors. Claim 3 implies that $H_1$ is a path on at most 5 vertices; further, $v$ must be adjacent to the endvertices of $H_1$. If $H_1$ has 2 vertices, then $G$ contains a 3-cycle, contradicting Claim 4. If $H_1$ has 3 vertices, then $G$ contains Figure 2(c), contradicting
Claim 2. If $H_1$ has 4 vertices, then we can delete it and extend the coloring of the smaller graph using one of the two extensions shown in Figure 3 depending on which colors are available at the 2-vertices (if the color used on the edge incident to the 3-vertex is seen by both 2-vertices, we use the extension on the left; otherwise, the extension on the right). Finally, suppose that $H_1$ has five vertices. Now $G$ is the Hajós join of $P^*$ and a smaller graph $J$; the copy of $P^* - e$ in $G$ consists of $H_1$, its adjacent 2-vertices, $v$, and $v$’s neighbor outside of $H_1$. Corollary 2 implies that $J$ is 3-critical. Since, $J$ has lower average degree than $G$, it contradicts our choice of $G$ as a minimal counterexample.

Figure 3: How to extend a coloring of $G \setminus H_1$ to $G$ when $H_1$ has order 4 and a 3-vertex $v$ has two neighbors in $H_1$.

Claim 5. The graph in Figure 4 cannot appear as a subgraph of $G$. Furthermore, the graph cannot appear as a subgraph even if one or more pairs of 2-vertices are identified. Thus, no 3-vertex has 3-neighbors in two distinct components of $H$, each of order at least 4.

The final statement follows immediately from the first two. We defer the proofs of those two statements to Lemma 3 in Section 3.

Recall now our outline of the discharging proof in the introduction. To begin, each 2-vertex takes some charge from each 3-neighbor. Now 3-vertices with 2-neighbors need more charge and 3-vertices with no 2-neighbors have extra charge. Thus, we call a 3-vertex with a 2-neighbor poor and a 3-vertex with no 2-neighbor rich. Roughly, we give charge from rich vertices to poor vertices. A rich vertex is type $(a, b, c)$ if it has 3-neighbors in components of

Figure 4: A configuration to forbid type $(\ast, 4^+, 4^+)$ vertices.
$H$ of orders $a, b, c$. Typically, we choose $a, b, c$ such that $a \leq b \leq c$. Analogous to vertices, a $t^+$-component (resp. $t^-$) has order at least (resp. at most) $t$. Claim 5 shows that vertices of type $(\ast, 4^+, 4^+)$ are forbidden. So Claims 3 and 4 imply that all rich vertices have type $(3^-, 3^-, 5^-)$. Now we can finish the proof by discharging. Recall that each vertex $v$ starts with charge $d(v)$. We apply the following three discharging rules, in succession.

(R1) Each 2-vertex takes charge $\frac{13}{37}$ from each neighbor.

(R2) Each rich vertex gives charge $\frac{1}{37}$ to each 3-neighbor in a $t$-component of $H$.

(R3) The 3-vertices in each component of $H$ average their charge.

Now we verify that each vertex finishes with charge at least $2 + \frac{26}{37}$.

Each 2-vertex finishes with charge $2 + 2(\frac{13}{37}) = 2 + \frac{26}{37}$.

Consider a component $H_1$ of $H$. After (R1), each vertex has charge $3 - \frac{13}{37}$. By (R2) and (R3), each vertex gains $2(\frac{1}{37})$, so finishes with charge $3 - \frac{13}{37} + 2(\frac{1}{37}) = 2 + \frac{26}{37}$.

Since type $(\ast, 4^+, 4^+)$ vertices are forbidden, each rich vertex gives away charge at most $(3 + 3 + 5)(\frac{1}{37})$, so finishes with charge at least $3 - \frac{11}{37} = 2 + \frac{26}{37}$. 

A common sentiment evoked by discharging proofs is that they’re easy to verify, but hard to find. So to shed some light on this process of discovery, we conclude this section with a synopsis of how we found the proof of the Main Theorem.

We began not knowing what edge bound we could prove. That specific value came last. The general idea was to get all vertices as much charge as possible, say charge $2 + \alpha$, for some $\alpha \in (\frac{2}{3}, \frac{3}{4})$. Since each 2-vertex needs charge $\alpha$, it takes charge $\frac{\alpha}{2}$ from each of its two 3-neighbors. Now 3-vertices with 2-neighbors have given away too much charge (thus, the name poor), so they need more charge from elsewhere. (If $\alpha = \frac{2}{3}$, then each 3-vertex gives away charge exactly $\frac{1}{3}$, so all vertices finish with charge $2 + \frac{2}{3}$, which proves Jakobsen’s bound: $|E(G)| \geq \frac{2}{3}|V(G)|$. Recall that Vizing’s Adjacency Lemma implies that each 3-vertex has at most one 2-neighbor.)

The poor vertices need extra charge, and they certainly won’t get it from other poor vertices. Thus, we must show that each poor vertex has some nearby rich vertex. This motivates our definition of $H$. So far we have used no reducible configurations (only Vizing’s Adjacency Lemma). By definition, each component of $H$ is either a path or a cycle. Each endpoint of each path of $H$ has a rich neighbor; crucially, rich vertices can share charge with the poor vertices in $H$ (thus, the name rich). However, any cycle component of $H$ has no such 3-neighbors to share charge with it. Thus, it is essential to show that $H$ contains no cycles. Furthermore, it is helpful to show that each path in $H$ is short, since the charge received by each path will be shared evenly among its vertices. To prove that $H$ has the desired structure, we introduce the reducible configurations shown in Figure 2(a) and Figure 2(b). It is at this point that we first encounter $P^*$. If some component of $H$ is a 6-cycle, then the entire graph $G$ is $P^*$; similarly if some component of $H$ is a 5-path, with its endpoints having a common neighbor, then $G$ is the Hajós join of $P^*$ and a smaller 3-critical graph. We also simplify things a bit by showing that no path of $H$ has endpoints with a common neighbor. The proof of this fact uses Figure 2(c).
To guarantee that each vertex of $H$ finishes with charge at least $2 + \alpha$, we split the charge given to each component of $H$ evenly among its vertices. Note that each path of $H$ gets charge from two vertices. Since each vertex of $H$ must reach final charge $2 + \alpha$, we have each $t$-component of $H$ take from each rich neighbor a charge proportional to $t$. This leads to the definition of type $(a, b, c)$ vertices. Of course, now we must bound the sum $a + b + c$ of a type $(a, b, c)$ vertex that is not reducible. This leads to the reducible configuration in Figure 4, which shows that when $a \leq b \leq c$ we may assume that $b \leq 3$. Each component of $H$ has order at most 5, so $a + b + c \leq 11$.

Following our framework above, each vertex must reach charge at least $2 + \alpha$, and each type $(a, b, c)$ vertex gives charge $t\beta$ to each adjacent $t$-component of $H$. Thus, we choose $\alpha$ and $\beta$ to maximize the minimum of the three expressions $2 + \alpha$, $3 - \frac{a}{2} + 2\beta$, and $3 - 11\beta$. This maximum is attained when the three quantities are equal, at $\alpha = \frac{26}{37}$ and $\beta = \frac{1}{37}$. This proves the bound $2|E(G)| \geq (2 + \frac{26}{37})|V(G)|$.

### 3 Reducibility

In this section we prove Claim 2 and Claim 5 from Section 2, that certain subgraphs $H$, not necessarily induced, are forbidden in $G$.

In each case, by criticality we 3-color all but the edges of $H$ (or some subgraph of it). If a color $w$ is used on an edge incident to a vertex $v_i$, then $v_i$ sees $w$. We want to show that we can always extend the partial coloring to all of $G$. We 3-color with the colors $x$, $y$, and $z$. Let $X = \{y, z\}$, $Y = \{x, z\}$, and $Z = \{x, y\}$. If a vertex $v_i$ in $H$ sees $x$, then the list of allowable colors for the uncolored edges incident to $v_i$ is $X$; similarly for colors $y$ and $z$. If the subgraph $H$ has $t$ vertices each with one incident colored edge, then we write this as an ordered $t$-tuple, where each entry is $X$, $Y$, or $Z$. We call each possible $t$-tuple a board. By permuting color classes, we will assume that the first coordinate in every board is $X$, and the first coordinate different is $Y$. By Polya counting, the number of boards is $(3^t - 1 + 1)/2$.

![Figure 5: A subgraph forbidden from appearing in 3-critical graph $G$.](image)

**Lemma 3.** The subgraph shown in Figure 5 (and in Figure 4) cannot appear in a 3-critical graph. Nor can it appear if we identify one or two vertex pairs in $\{v_6, v_7, v_{14}, v_{15}\}$.

**Proof.** We first consider the case where no pairs of 2-vertices are identified. Note that the right and left sides of the figure are symmetric. By criticality, construct a partial 3-coloring
of all of $G$ except the edges incident to $v_1$, $v_3$, and $v_4$. Since $t = 4$ (as defined in introduction to this section), we have $(3^3 + 1)/2 = 14$ boards. We begin by showing that for 12 of these 14 boards, we can extend the coloring to all of $G$.

If $v_2$ and $v_5$ see distinct colors, then Figure 6(a) shows how to extend the coloring unless the board is $(X, X, Y, Y)$: simply color greedily along the path of uncolored edges, starting at $v_6$ and ending at $v_1$. Now suppose instead that $v_2$ and $v_5$ see the same color, $x$. If $v_6$ or $v_7$ sees $x$, then we can extend the coloring as in Figure 6(b): now color greedily along the path of uncolored edges, ending at $v_7$. Further, if $v_6$ and $v_7$ see distinct colors, then we can color as in Figure 6(c). Thus, we conclude that we can extend the partial coloring to $G$ unless the board is either $(X, X, Y, Y)$ or $(X, Y, Y, X)$. Note that these two bad boards differ in the colors used on two edges, even up to all permutations of color classes. Thus, if at least one pendant edges is not yet colored, we can always find an extension of the partial coloring.

Now suppose that $G$ contains a copy of Figure 5. By criticality, we get a 3-coloring of all of $G$ except the edges with both endpoints in Figure 5. Our goal is to color the two remaining edges incident to $v_8$ so that both the left side and the right side can be colored from their resulting boards. As shown above, we must color $v_5v_8$ and $v_8v_{10}$ so that neither the left or right board is $(X, X, Y, Y)$ or $(X, Y, Y, X)$.

Given the colors incident to $v_2$, $v_6$, and $v_7$, at most one choice of color for $v_5v_8$ gives a bad board for the left side. Similarly, at most one choice of color for $v_8v_{10}$ is bad for the right side. We can color the edges as desired unless the color that is bad on $v_5v_8$ for the left side is the same as the color that is bad on $v_8v_{10}$ for the right side, and that color, say $x$, is different from the color $y$ seen by $v_8$. So suppose this is true. Now perform an $(x, y)$-Kempe swap at $v_8$. If this Kempe chain ends at neither the left nor right side, then we color $v_5v_8$ and $v_8v_{10}$ arbitrarily. Now we can color each side. So suppose instead that the Kempe chain ends at the left side (by symmetry). Now we can color the left side, since $v_5v_8$ is uncolored. Afterward, the color for $v_8v_{10}$ is determined, and we can color the right side. This completes the case where no pairs of 2-vertices are identified.

Now we consider the case where two vertex pairs in $\{v_6, v_7, v_{14}, v_{15}\}$ are identified. Since $G$ has no 3-cycles, each of $v_6, v_7$ must be identified with one of $v_{14}, v_{15}$. By criticality, color all edges except those with both endpoints in Figure 5. We have only 3 incident colored
using Figure 6.

For board \((X, X, X)\), color \(v_5v_8\) and \(v_4v_7\) with \(y\) and color \(v_8v_{10}\) and \(v_{11}v_{14}\) with \(z\). (Perhaps \(v_7 = v_{14}\), but this is okay.) Now we can extend the coloring to each side, as in Figure 6(a). A similar strategy works in every case except \((X, Y, X)\). We always color \(v_5v_8\) and \(v_8v_{10}\) so that their colors differ from those seen by \(v_2\) and \(v_{13}\), respectively. Next, we color \(v_4v_7\) and \(v_{11}v_{14}\) to match \(v_8v_9\) and \(v_8v_{10}\), respectively. Finally, we can color each side as in Figure 6(a). So consider case \((X, Y, X)\). Color \(v_5v_8\) with \(x\) and \(v_8v_{10}\) with \(z\). Now color \(v_3v_4\) with \(x\) and \(v_{14}v_i\) with \(y\), where \(i \in \{3, 4\}\). Since \(v_{10}\) and \(v_{14}\) see different colors, we extend the right side as in Figure 6(a). We extend the left side as in Figure 6(b). This completes the case of two pairs of identified vertices.

Now suppose that one vertex pair in \(\{v_6, v_7, v_{14}, v_{15}\}\) is identified; we consider three cases. The identified pair is either \((v_6, v_{15})\), \((v_7, v_{14})\), or \((v_6, v_{14})\); we call these cases “outside”, “inside”, and “mixed”. In each case, five vertices see colors, but we initially consider only the colors seen by \(v_2\), \(v_8\), and \(v_{13}\). Thus, for example, we write the board \((X, Y, Z)\) to signify that \(v_2\) sees \(x\), \(v_8\) sees \(y\), and \(v_{13}\) sees \(z\).

First consider outside. Suppose we have board \((X, X, Y)\). Color \(v_3v_6\) with \(x\), color \(v_5v_8\) and \(v_{12}v_{15}\) with \(y\), and color \(v_8v_{10}\) with \(z\). We can extend the coloring on each side as in Figure 6(a). A similar strategy works for boards \((X, Y, Y)\) and \((X, Y, Z)\). Consider instead \((X, X, X)\). Now color \(v_5v_8\) and \(v_{12}v_{15}\) with \(x\) and color \(v_8v_{10}\) with \(z\). We can color the right as in Figure 6(a) and the left as in Figure 6(b). Finally, consider \((X, X, X)\). If \(v_7\) sees \(x\), then color \(v_{12}v_{15}\) with \(x\), color \(v_5v_8\) with \(y\), and color \(v_8v_{10}\) with \(z\). Now color both sides as in Figure 6(a). Otherwise, by symmetry \(v_7\) sees \(y\). Now color \(v_{12}v_{15}\) with \(x\), \(v_8v_{10}\) with \(y\), and \(v_5v_8\) with \(z\). We can again color both sides as in Figure 6(a).

Now consider inside. This case is similar to above. Consider a board other than \((X, Y, X)\). Color \(v_5v_8\) and \(v_8v_{10}\) to differ from the colors seen by \(v_2\) and \(v_{13}\), respectively. Now color \(v_4v_7\) and \(v_{11}v_{14}\) to match \(v_8v_9\) and \(v_8v_{10}\), respectively. Finally, color each side as in Figure 6(a). So consider \((X, Y, X)\). If \(v_6\) sees \(x\), then color \(v_5v_8\) with \(x\) and \(v_8v_{10}\) with \(z\). Now we can color the right first, then color the left, since the left board is \((X, X, *, X)\). So \(v_6\) does not see \(x\). Now color \(v_5v_8\) with \(z\) and \(v_8v_{10}\) with \(x\). Color the right first, then color left as in Figure 6(a), since \(v_2\) and \(v_6\) see distinct colors.

Finally, consider mixed. Recall that \(v_6\) and \(v_{14}\) are identified. Consider a board other than \((X, X, X)\) and \((X, Y, Y)\). If \(v_{13}\) and \(v_{15}\) see distinct colors, then color \(v_8v_{10}\) with a color not seen by \(v_{13}\). Now color the left side, then extend to the right as in Figure 6(a). Otherwise \(v_{13}\) and \(v_{15}\) see the same color, so use that color on \(v_8v_{10}\). Now color the left, then extend to the right, as in Figure 6(b). Instead, consider \((X, X, X)\). Color \(v_3v_6\) with \(x\), color \(v_5v_8\) with \(y\), and both \(v_8v_{10}\) and \(v_{11}v_{14}\) with \(z\). Now extend both sides as in Figure 6(a). Finally, consider \((X, Y, Y)\). If \(v_{15}\) sees a color other than \(y\), then color \(v_8v_{10}\) to avoid the color seen by \(v_{15}\). Now color the left, followed by the right, as in Figure 6(a). Similarly, if \(v_7\) sees \(x\), then color \(v_5v_8\) with \(x\) and color the right, followed by the left. Likewise, if \(v_7\) sees \(y\), then color \(v_5v_8\) with \(z\) and color the right, followed by the left. Thus, we conclude that \(v_7\) sees \(z\) and \(v_{15}\) sees \(y\). Now perform an \((x, y)\)-Kempe swap at \(v_8\). The resulting board will be one of the cases above.
Lemma 4. The subgraph in Figure 7 (and Figure 2(a)) cannot appear in a 3-critical graph.

Proof. To describe the boards, we use an ordered 6-tuple, where coordinate \( i \) is the list of colors missed by \( v_i \). By Polya counting, the number of boards is \((3^5 + 1)/2 = 122\). The number of these of type \((X, *, *, *, *, X)\) is \((3^4 + 1)/2 = 41\). In the remaining \(122 - 41 = 81\) boards, the first and last coordinates differ. By symmetry of color classes, we denote them as type \((X, *, *, *, *, Y)\). Similar to the previous proof, we seek to color edge \( v_7v_8 \), so that the resulting boards for the left and right side are both colorable. (Some example colorings are shown in Figure 8.) However, some boards are not immediately colorable. Thus, we sometimes perform one or more Kempe swaps on a board before the result is colorable. If, from a given board, we can always perform some sequence of Kempe swaps to reach a colorable board, then we win on that board. Suppose that vertices \( v_i \) and \( v_j \) each see exactly one of colors \( x \) and \( y \). If \( v_i \) and \( v_j \) lie in the same \((x, y)\)-Kempe chain, then they are \((x, y)\)-paired. If a vertex \( v_i \) is not \((x, y)\)-paired with any vertex \( v_j \), then it is \((x, y)\)-unpaired. The definitions for color pairs \((x, z)\) and \((y, z)\) are analogous. To prove that we can win on all boards, we consider separately type \((X, *, *, *, *, X)\) and type \((X, *, *, *, *, Y)\).

Case 1: The board has type \((X, *, *, *, *, X)\). We group boards based on how many leading \( X \)s they have. This gives five possibilities: \((X, X, X, X, X, X)\); \((X, X, X, X, Y, X)\); \((X, X, X, Y, *, X)\); \((X, X, Y, *, *, X)\); and \((X, Y, *, *, *, X)\). The first two types can be colored using two copies of Figure 8(a). Two instances of the third type can be colored again using two copies of Figure 8(a). The remaining instance of this type is not colorable: \((X, X, Y, Z, X)\). Among type \((X, X, Y, *, *, X)\), seven are colorable using Figure 8(a) and either Figure 8(a) or Figure 8(b). This leaves \((X, X, Y, Z, X)\) and \((X, X, Y, Z, Y, X)\). The first is colorable using Figure 8(e) and a reflected copy of Figure 8(e). The second using Figure 8(f) and a reflected copy of Figure 8(f).

Now we consider type \((X, Y, *, *, *, X)\). Of these 27 boards, seven of type \((X, Y, X, *, *, X)\) are colorable using Figure 8(a) and either Figure 8(a) or Figure 8(b). Similarly, seven of type \((X, Y, Y, *, *, X)\). Also, \((X, Y, Z, Y, X)\) is colorable using Figure 8(f) and a reflected copy of Figure 8(f). Four of type \((X, Y, Z, *, *, X)\) are colorable using Figure 8(f) and a reflected copy of Figure 8(f). Combining these eight uncolorable boards with the one in the previous paragraph, we conclude that type \((X, *, *, *, *, X)\) is colorable unless it is one of the following nine boards: \((X, X, Y, Z, X)\); \((X, Y, X, Y, Z, X)\); \((X, Y, Z, X, Y, X)\); \((X, Y, Z, Y, Z, X)\); \((X, Y, Z, Z, X, Y)\); \((X, Y, Z, Z, Z, X)\); \((X, Y, Z, Y, Z, X)\); \((X, Y, Z, Z, Z, X)\).

Consider the eighth of these: \((X, Y, Z, Y, Z, X)\). Now we \((y, z)\)-swap at \( v_2 \) (regardless of its
(y, z)-pair. After permuting color classes, the result is (X, Y, Z, Z, X) or (X, Y, Y, Y, X) or (X, Y, Y, Z, X, X). Each validity coloring of the configuration (for some choice of board) consists of two figures that share a color on the edge where they overlap. For example, the board (X, Y, Z, X, Y, Y) can be colored using Figures 8(f) on the left and 8(g) on the right.

(72x348)-(y,z)-pair). After permuting color classes, the result is (X, Y, Z, Z, Y, X) or (X, Y, Y, Y, Y, X) or (X, Y, Y, Z, Z, X, X). Each validity coloring of the configuration (for some choice of board) consists of two figures that share a color on the edge where they overlap. For example, the board (X, Y, Z, X, Y, Y) can be colored using Figures 8(f) on the left and 8(g) on the right.

We first consider board (X, Y, X, Z, Y, Z, X, Y, Y, X). If (v2, v4) or (v4, v3) is a (x,y)-pair, then the resulting board is colorable, as shown above; similarly if v4 is (x,y)-unpaired, then we reach a colorable board. Otherwise, (v4, v5) is a (y, z)-pair, so we reach (X, Y, X, Z, Y, X, Y, Y, Y, Y, Y, X, X), the second. A similar argument shows that we can reach the third from the second. Thus it suffices to show that we can win on boards (X, X, X, Y, Z, X), (X, Y, X, Y, X, Z, Z, X, Y, Y, Y, Y, X, X), and (X, Y, Y, Z, X, X, Y, Y, Y, Y, X, X). We next consider board (X, X, X, Y, Z, X). We (x,z)-swap v3 if either v3 is (x, z)-unpaired or (v2, v3) or (v3, v5) is an (x, z)-pair. In each case, the result is colorable, as shown
above. So assume that either \((v_3, v_6)\) or \((v_3, v_1)\) is an \((x, z)\)-pair. In the first case, we \((x, z)\)-swap \((v_3, v_6)\) to get \((X, X, Y, Z, Y, Y)\) (after permuting color classes). This board is colorable using Figure 8(e) and Figure 8(d). So assume that \((v_3, v_1)\) is an \((x, z)\)-pair. Now consider the \((x, z)\)-pair of \(v_5\). If \(v_5\) is \((x, z)\)-unpaired, then we \((x, z)\)-swap \(v_5\) to get \((X, X, X, Y, X, X)\), which is colorable using two copies of Figure 8(a). If \((v_5, v_2)\) is an \((x, z)\)-pair, then we \((x, z)\)-swap \((v_5, v_2)\) to get \((X, Y, Z, X, X)\) (up to symmetry), which is colorable using Figure 8(a) and Figure 8(b). So \((v_5, v_4)\) must be an \((x, z)\)-pair, which means that \(v_2\) is \((x, z)\)-unpaired. So we \((x, z)\)-swap \(v_2\) to get \((X, Y, Z, Y, X)\) (up to symmetry). As shown above, this board is equivalent to \((X, Y, X, Z, X)\), which we showed in the previous paragraph is a win.

Finally, we consider board \((X, Y, Y, Z, X)\). Now we \((x, y)\)-swap \(v_4\), as long as \((v_4, v_1)\) is not an \((x, y)\)-pair. If \(v_4\) is \((x, y)\)-unpaired, this gives \((X, Y, Y, Z, X)\); if \((v_2, v_4)\) is a \((x, y)\)-pair, this gives \((X, X, Y, Z, X)\); if \((v_3, v_4)\) is a \((x, y)\)-pair, this gives \((X, Y, X, Z, X)\). Each of these boards is colorable using Figure 8(a) and Figure 8(b). If \((v_4, v_6)\) is an \((x, y)\)-pair, then we \((x, y)\)-swap \((v_4, v_6)\) to get \((X, Y, Y, X, Z, Y)\), which is colorable using Figure 8(a) and Figure 8(e). So assume that \((v_1, v_4)\) is an \((x, y)\)-pair. If \((v_2, v_3)\) is also an \((x, y)\)-pair, then we \((x, y)\)-swap there to get \((X, X, Y, Z, X)\), on which we win, by the previous paragraph. If \(v_2\) has no \((x, y)\)-pair, then we \((x, y)\)-swap at \(v_2\) to get \((X, Y, Y, Z, X)\), which is colorable using Figure 8(e) and a reflected copy of Figure 8(e). Thus \((v_2, v_6)\) is an \((x, y)\)-pair. So \(v_3\) is \((x, y)\) unpaired; now \((x, y)\)-swapping at \(v_3\) gives \((X, Y, X, Y, Z, X)\), on which we win, as shown above. Hence, we win on all boards of type \((X, *, *, *, *, Y)\).

**Case 2: The board has type \((X, *, *, *, *, Y)\).** We first show that we can win all boards of type \((X, X, Y, *, *, Y)\). In fact, all such boards are colorable. Seven of the nine colorings use Figure 8(e) with either Figure 8(c) or Figure 8(d). One of the remaining colorings uses Figure 8(a) and Figure 8(e), the other uses Figure 8(f) and Figure 8(g). An analogous argument (with Figure 8(b) in place of Figure 8(a)) shows that we win all boards of type \((X, X, *, *, *, Y)\).

Now consider boards of type \((X, X, X, *, *, Y)\). Four of these are colorable using Figure 8(a) and Figure 8(e). So we consider five boards: \((X, X, X, X, Y)\); \((X, X, X, Y, Y)\); \((X, X, Y, X, Y)\); \((X, X, Y, Y, Y)\); \((X, X, X, Z, X, Y)\); \((X, X, X, Z, Y, Y)\); \((X, X, Z, X, Y)\); \((X, X, Z, Y, Y)\). In each case, \((v_1, v_6)\) must be an \((x, y)\)-pair; otherwise, we \((x, y)\)-swap at \(v_6\) and win by Case 1. Similarly, \((v_2, v_3)\) must be an \((x, y)\)-pair; otherwise, we \((x, y)\)-swap at \(v_3\), and reduce to a board of type \((X, X, Y, *, *, Y)\), which we win as above. In the fourth and fifth cases, we \((x, y)\)-swap \(v_4\) and \(v_5\), respectively; in each case, the resulting board can be colored using Figure 8(a) and Figure 8(e). Hence, we are in one of the first three cases, so \((v_4, v_5)\) is an \((x, y)\)-pair. In the second case, we \((x, y)\)-swap \((v_4, v_5)\); this yields \((X, X, X, Y, Z, Y)\), which is colorable using Figure 8(a) and Figure 8(e). In the first and third cases, \((x, y)\)-swap \((v_2, v_3)\) and possibly \((v_4, v_5)\), in each case yielding \((X, Y, X, X, Y)\). This is colorable using Figure 8(f) and Figure 8(g). Thus, we win all boards of type \((X, X, *, *, *, Y)\).

Now consider a board of type \((X, Y, *, *, *, Y)\). We can assume that \((v_1, v_6)\) is an \((x, y)\)-pair, for otherwise we \((x, y)\)-swap at \(v_6\) and can win by Case 1. However, now we \((x, y)\)-swap \(v_2\). This yields a board of type \((X, X, *, *, *, Y)\), which we can win, as shown above. Thus, we win all boards of type \((X, Y, *, *, *, Y)\).

Now we consider boards of type \((X, Z, *, *, *, Y)\). First consider type \((X, Z, *, *, *, Y)\). Seven of these nine boards are colorable using Figure 8(e) with either Figure 8(c) or Fig-
Figure 8(d). This leaves \((X, Z, Z, X, Z, Y)\) and \((X, Z, Z, X, Z, Y)\). The first is colorable using Figure 8(b) and Figure 8(e), so consider the second. Now \((v_1, v_2)\) must be an \((x, z)\)-pair; otherwise, we \((x, z)\)-swap at \(v_2\) and reduce to type \((X, X, *, *, *, Y)\). If either \(v_4\) or \(v_5\) is \((x, z)\)-unpaired, then we \((x, z)\)-swap there and reduce to a colorable board. Hence, \((v_4, v_5)\) is a \((x, z)\)-pair. Now, again, we \((x, z)\)-swap \((v_4, v_5)\) and reduce to \((X, Z, Z, X, Z, Y)\), which is colorable. Thus, we win all boards of type \((X, Z, Z, *, *, Y)\).

Now consider type \((X, Z, Y, *, *, Y)\). Note that \((v_2, v_6)\) must be a \((y, z)\)-pair; otherwise, we \((y, z)\)-swap \(v_2\) and reduce to the case \((X, Y, *, *, *, Y)\). However, now we \((y, z)\)-swap \(v_3\), and reduce to the case \((X, Z, Z, *, *, Y)\).

Finally, consider type \((X, Z, X, *, *, Y)\). Recall that \((v_1, v_6)\) must be an \((x, y)\)-pair; otherwise we \((x, y)\)-swap at \(v_6\) and win by Case 1. Now we \((x, y)\)-swap at \(v_3\), and win by the previous paragraph. So we win all boards of type \((X, Z, *, *, *, Y)\). Thus, we win all boards of type \((X, *, *, *, *, Y)\), which completes the proof of both the claim and the lemma. □

4 An Improved Bound using a Computer

![Figure 9: Extra reducible configurations.](image_url)

The reducible configurations in this paper were originally found by computer. The computer uses an abstract definition capturing the notion of “colorable after performing some Kempe swaps”, which frees it from considering an embedding in an ambient critical graph. This is called fixability and extends the idea in [6] from stars to arbitrary graphs. The computer is able to prove many reducibility results for which we have yet to find short proofs. Here we show how to use some of these reducible configurations to further improve the bound on the average degree of 3-critical graphs. We give a larger survey of what can be proved with these computer results in [1].

14
Lemma 5. The configurations in Figure 9 cannot be subgraphs of a 3-critical graph. In particular, all rich vertices are of type $(2^-, 2^-, 5^-)$.

Proof. Suppose that $v$ is a rich vertex of type $(\ast, 3^+, 3^+)$. If these two adjacent components of $H$ have no common 2-neighbors, then $G$ has a copy of Figure 9(i). Otherwise, the components share one or more common 2-neighbors, and $G$ contains one of Figure 9(a)–(h). The computer is able to generate proofs in \LaTeX, but at about 100 pages this one is not a fun read: https://dl.dropboxusercontent.com/u/8609833/Papers/big%20tree.pdf

Theorem 6. If a graph $G$ with $\Delta = 3$ is critical, then either $2|E(G)| \geq (2 + \frac{22}{31})|V(G)|$ or else either $G$ is $P^*$ or $J_1$ (Woodall’s first example).

Proof. By Lemma 5, we have $a + b + c \leq 9$ for every rich vertex of type $(a, b, c)$. So, we obtain our desired bound when the three quantities $2 + \alpha$, $3 - \frac{\alpha}{2} + 2\beta$ and $3 - 9\beta$ are equal. This happens when $\alpha = \frac{22}{31}$ and $\beta = \frac{1}{31}$, which yields $2|E(G)| \geq (2 + \frac{22}{31})|V(G)|$.

References


