

# Planar graphs have independence ratio at least $\frac{3}{13}$

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Submitted: June 6, 2015; Accepted: June 25, 2016; Published: XX

Mathematics Subject Classifications: 05C69, 05C10

## Abstract

The 4 Color Theorem (4CT) implies that every  $n$ -vertex planar graph has an independent set of size at least  $\frac{n}{4}$ ; this is best possible, as shown by the disjoint union of many copies of  $K_4$ . In 1968, Erdős asked whether this bound on independence number could be proved more easily than the full 4CT. In 1976 Albertson showed (independently of the 4CT) that every  $n$ -vertex planar graph has an independent set of size at least  $\frac{2n}{9}$ . Until now, this remained the best bound independent of the 4CT. Our main result improves this bound to  $\frac{3n}{13}$ .

## 1 Introduction

An *independent set* is a subset of vertices that induce no edges. The independence number  $\alpha(G)$  of a graph  $G$  is the size of a largest independent set in  $G$ . Determining the independence number of an arbitrary graph  $G$  is widely-studied and well-known to be NP-complete. In fact, this problem remains NP-complete, even when restricted to planar graphs of maximum degree 3 (see, for example, [5, Lemma 1]). Thus, much work in this area focuses on proving lower bounds for the independence number of some special class of graphs, often in terms of  $|V(G)|$ . The *independence ratio* of a graph  $G$  is the quantity  $\frac{\alpha(G)}{|V(G)|}$ .

An immediate consequence of the 4 Color Theorem [2, 3] is that every planar graph has independence ratio at least  $\frac{1}{4}$ ; simply take the largest color class. In fact, this bound is best possible, as shown by the disjoint union of many copies of  $K_4$ . In 1968, Erdős [4] suggested that perhaps this corollary could be proved more easily than the full 4 Color Theorem. And in 1976, Albertson [1] showed (independently of the 4 Color Theorem) that every planar graph has independence ratio at least  $\frac{2}{9}$ . Our main theorem improves this bound to  $\frac{3}{13}$ .

**Theorem 1.** *Every planar graph has independence ratio at least  $\frac{3}{13}$ .*

The proof of Theorem 1 is heavily influenced by Albertson's proof. One apparent difference is that our proof uses the discharging method, while his does not. However, this distinction is largely cosmetic. To demonstrate this point, we begin with a short discharging version of the final step in Albertson's proof, which he verified using edge-counting. Although the arguments are essentially equivalent, the discharging method is somewhat more flexible. In part it was this added flexibility that allowed us to push his ideas further.

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The proof of our main result has the following outline. The bulk of the work consists in showing that certain configurations are *reducible*, i.e., they cannot appear in a minimal counterexample to the theorem. The remainder of the proof is a counting argument (called *discharging*), where we show that every planar graph contains one of the forbidden configurations; hence, it is not a minimal counterexample.

In the discharging section, we give each vertex  $v$  initial charge  $d(v) - 6$ , where  $d(v)$  is the degree of  $v$ . By Euler's formula the sum of the initial charges is  $-12$ . Our goal is to redistribute charge, without changing the sum, (assuming that  $G$  contains no reducible configuration) so that every vertex finishes with nonnegative charge. This contradiction proves that, in fact,  $G$  must contain a reducible configuration. To this end, we want to show that  $G$  contains a reducible configuration whenever it has many vertices of degree at most 6 near each other, since vertices of degree 5 will need to receive charge and vertices of degree 6 will have no spare charge to give away. (We will see in Lemma 6 that  $G$  must have minimum degree 5.) Most of the work in the reducibility section goes into proving various formalizations of this intuition.

Typically, proofs like ours present the reducibility portion before the discharging portion. However, because many of our reducibility arguments are quite technical, we make the unusual choice to give the discharging first, with the goal of providing context for the reducible configurations. (Usually the process of finding a proof switches back and forth between discharging and reducibility. By necessity, though, the proof must present one of these first.)

We start with definitions. A  $k$ -vertex is a vertex of degree  $k$ ; similarly, a  $k^-$ -vertex (resp.  $k^+$ -vertex) has degree at most (resp. at least)  $k$ . A  $k$ -neighbor of a vertex  $v$  is a  $k$ -vertex that is a neighbor of  $v$ ; and  $k^-$ -neighbors and  $k^+$ -neighbors are defined analogously. A  $k$ -cycle is a cycle of length  $k$ . A vertex set  $V_1$  in a connected graph  $G$  is *separating* if  $G \setminus V_1$  has at least two components. A cycle  $C$  is separating if  $V(C)$  is separating. An *independent  $k$ -set* is an independent set of size  $k$ . When vertices  $u$  and  $v$  are adjacent, we write  $u \leftrightarrow v$ ; otherwise  $u \not\leftrightarrow v$ .

For a vertex  $v$ , let  $H_v$  denote the subgraph induced by the 5-neighbors and 6-neighbors of  $v$ . Throughout the proof we consider a (hypothetical) minimal counterexample  $G$ , which will be a triangulation. In Lemma 2, we show that  $G$  has no separating 3-cycle. These properties together imply that, for every vertex  $v$ , the subgraph induced by the neighbors of  $v$  is a cycle. If some  $w \in V(H_v)$  has  $d_{H_v}(w) = 0$ , then  $w$  is an *isolated* neighbor of  $v$ ; otherwise  $w$  is a *non-isolated* neighbor. A non-isolated 5-neighbor of a vertex  $v$  is *crowded* (with respect to  $v$ ) if it has two 6-neighbors in  $H_v$ . We use crowded 5-neighbors in the discharging proof to help ensure that 7-vertices finish with sufficient charge, specifically to handle the configuration in Figure 1.

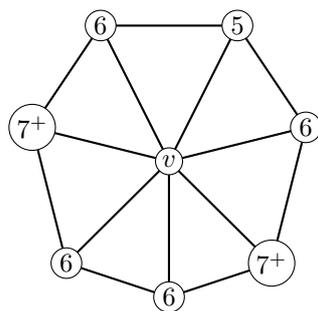


Figure 1: A 7-vertex  $v$  gives no charge to any crowded 5-neighbor.

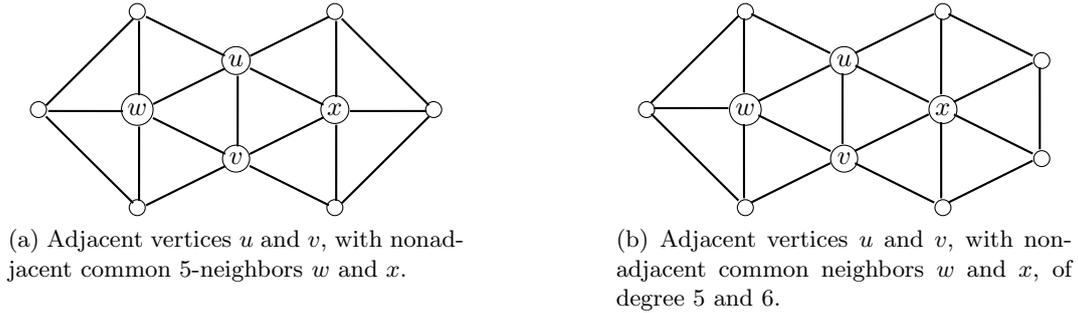


Figure 2: The two instances of configuration  $H$ .

## 2 Discharging: A Warmup

As a warmup to our main proof, in this section we give a short discharging proof that every planar triangulation with minimum degree 5 and no separating 3-cycle must contain a certain configuration, which Albertson showed could not appear in a minimal planar graph with independence ratio less than  $\frac{2}{9}$ . (In fact, finding this proof helped encourage us to begin work on the present paper.)

**Lemma A.** *Let  $u$  and  $v$  be adjacent vertices, such that  $uvw$  and  $uvx$  are 3-faces and  $d(w) = 5$  and  $d(x) \leq 6$ ; call this configuration  $H$ . (See Figure 2.) If  $G$  is a plane triangulation with minimum degree 5 and no separating 3-cycle, then  $G$  contains a copy of  $H$ .*

*Proof.* Assume that  $G$  has minimum degree 5 and no separating 3-cycle, but also has no copy of  $H$ . This assumption leads to a contradiction, which implies the result. An immediate consequence of this assumption (by Pigeonhole) is that the number of 5-neighbors of each vertex  $v$  is at most  $\lfloor \frac{d(v)}{2} \rfloor$ . Below, when we verify that each vertex finishes with nonnegative charge, we consider both the degree of  $v$  and its number of 5-neighbors. We write  $(a, b)$ -vertex to denote a vertex of degree  $a$  that has  $b$  5-neighbors.

We assign to each vertex  $v$  a charge  $\text{ch}(v)$ , where  $\text{ch}(v) = d(v) - 6$ . Note that  $\sum_{v \in V} \text{ch}(v) = 2|E(G)| - 6|V(G)|$ . Since  $G$  is a plane triangulation, Euler's formula implies that  $2|E(G)| - 6|V(G)| = -12$ . Now we redistribute the charge, without changing the sum, so that each vertex finishes with nonnegative charge. This redistribution is called *discharging*, and we write  $\text{ch}^*(v)$  to denote the charge at each vertex  $v$  after discharging. Since each vertex finishes with nonnegative charge, we get the obvious contradiction  $-12 = \sum_{v \in V} \text{ch}(v) = \sum_{v \in V} \text{ch}^*(v) \geq 0$ . We redistribute the charge via the following three discharging rules, which we apply simultaneously everywhere they are applicable.

- (R1) Each  $7^+$ -vertex gives charge  $\frac{1}{3}$  to each 5-neighbor.
- (R2) Each  $7^+$ -vertex gives charge  $\frac{1}{7}$  to each 6-neighbor that has at least one 5-neighbor.
- (R3) Each 6-vertex gives charge  $\frac{2}{7}$  to each 5-neighbor.

We now verify that after discharging, each vertex  $v$  has nonnegative charge. We repeatedly use that  $G$  has no copy of configuration  $H$ . In particular, this implies that the number of 5-neighbors for each vertex  $v$  is at most  $\frac{d(v)}{2}$ .

**d(v) = 5:** Each (5, 0)-vertex  $v$  has five  $6^+$ -neighbors, so  $\text{ch}^*(v) \geq -1 + 5\left(\frac{2}{7}\right) > 0$ . Each (5, 1)-vertex  $v$  has four  $6^+$ -neighbors, at least two of which are  $7^+$ -neighbors; so  $\text{ch}^*(v) \geq -1 + 2\left(\frac{1}{3}\right) + 2\left(\frac{2}{7}\right) > 0$ . Each (5, 2)-vertex  $v$  has three  $7^+$ -neighbors (otherwise  $G$  contains a copy of  $H$ ), so  $\text{ch}^*(v) = -1 + 3\left(\frac{1}{3}\right) = 0$ .

**d(v) = 6:** Each (6, 0)-vertex  $v$  has  $\text{ch}^*(v) = \text{ch}(v) = 0$ . Each (6, 1)-vertex  $v$  has at least two  $7^+$ -neighbors, so  $\text{ch}^*(v) \geq 0 + 2\left(\frac{1}{7}\right) - \left(\frac{2}{7}\right) = 0$ . Each (6, 2)-vertex  $v$  has four  $7^+$ -neighbors, so  $\text{ch}^*(v) = 0 + 4\left(\frac{1}{7}\right) - 2\left(\frac{2}{7}\right) = 0$ .

**d(v) = 7:** Each (7, 0)-vertex  $v$  has  $\text{ch}^*(v) \geq 1 - 7\left(\frac{1}{7}\right) = 0$ . Each (7, 1)-vertex  $v$  has six  $6^+$ -neighbors, at least two of which are  $7^+$ -vertices (namely, the neighbors that are two further clockwise and two further counterclockwise around  $v$  from the 5-vertex; otherwise  $G$  has a copy of  $H$ ). So  $\text{ch}^*(v) \geq 1 - 1\left(\frac{1}{3}\right) - 4\left(\frac{1}{7}\right) > 0$ . Each (7, 2)-vertex has five  $6^+$ -neighbors, at least three of which are  $7^+$ -vertices; so  $\text{ch}^*(v) \geq 1 - 2\left(\frac{1}{3}\right) - 2\left(\frac{1}{7}\right) > 0$ . Each (7, 3)-vertex has four  $7^+$ -neighbors, so  $\text{ch}^*(v) = 1 - 3\left(\frac{1}{3}\right) = 0$ .

**d(v) = 8:** Now  $v$  has at most four 5-neighbors, and gives each of these charge  $\frac{1}{3}$ ; also  $v$  gives each other neighbor charge at most  $\frac{1}{7}$ . Thus  $\text{ch}^*(v) \geq 8 - 6 - 4\left(\frac{1}{3}\right) - 4\left(\frac{1}{7}\right) > 0$ .

**d(v)  $\geq$  9:** Now  $v$  gives each neighbor charge at most  $\frac{1}{3}$ , so  $\text{ch}^*(v) \geq d(v) - 6 - d(v)\left(\frac{1}{3}\right) = \frac{2}{3}(d(v) - 9) \geq 0$ .

Thus  $-12 = \sum_{v \in V} \text{ch}(v) = \sum_{v \in V} \text{ch}^*(v) \geq 0$ . This contradiction implies the result.  $\square$

### 3 Discharging

In this section we present the discharging argument for the proof of Theorem 1. It is convenient to collect all of the reducibility lemmas that we use to analyze the discharging (but prove later).

**Lemma 8.** *Every independent set  $J$  in a minimal  $G$  with  $|J| = 2$ , satisfies  $|N(J)| \geq 9$ .*

**Lemma 9.** *A minimal  $G$  cannot have two nonadjacent 5-vertices with at least two common neighbors. In particular, each vertex  $v$  in  $G$  has  $\frac{1}{2}d(v)$  or more  $6^+$ -neighbors.*

**Lemma 17.** *Every minimal  $G$  has no 6-vertex  $v$  with  $6^-$ -neighbors  $u_1$ ,  $u_2$ , and  $u_3$  that are pairwise nonadjacent.*

**Lemma 18.** *Every minimal  $G$  has no 6-vertex  $v$  with pairwise nonadjacent neighbors  $u_1$ ,  $u_2$ , and  $u_3$ , where  $d(u_1) = 5$ ,  $d(u_2) \leq 6$ , and  $d(u_3) = 7$ .*

**Lemma 19.** *Let  $u_1$  be a 6-vertex with nonadjacent vertices  $u_2$  and  $u_3$  each at distance two from  $u_1$ , where  $u_2$  is a 5-vertex and  $u_3$  is a  $6^-$ -vertex. A minimal  $G$  cannot have  $u_1$  and  $u_2$  with two common neighbors, and also  $u_1$  and  $u_3$  with two common neighbors.*

**Lemma 20.** *Every minimal  $G$  has no 7-vertex  $v$  with a 5-neighbor and two other  $6^-$ -neighbors,  $u_1$ ,  $u_2$ , and  $u_3$ , that are pairwise nonadjacent.*

**Lemma 21.** *Let  $v_1, v_2, v_3$  be the corners of a 3-face, each a  $6^+$ -vertex. Let  $u_1, u_2, u_3$  be the other pairwise common neighbors of  $v_1, v_2, v_3$ , i.e.,  $u_1$  is adjacent to  $v_1$  and  $v_2$ ,  $u_2$  is adjacent to  $v_2$  and  $v_3$ , and  $u_3$  is adjacent to  $v_3$  and  $v_1$ . We cannot have  $|N(\{u_1, u_2, u_3\})| \leq 13$ . In particular, we cannot have  $d(u_1) = d(u_2) = 5$  and  $d(u_3) \leq 6$ .*

**Lemma 22.** *Let  $u_1$  be a 7-vertex with nonadjacent 5-vertices  $u_2$  and  $u_3$  each at distance two from  $u_1$ . A minimal  $G$  cannot have  $u_1$  and  $u_2$  with two common neighbors and also  $u_1$  and  $u_3$  with two common neighbors.*

**Lemma 23.** *Suppose that a minimal  $G$  contains a 7-vertex  $v$  with no 5-neighbor. Now  $v$  cannot have at least five 6-neighbors, each of which has a 5-neighbor.*

**Theorem 1.** *Every planar graph  $G$  has independence ratio at least  $\frac{3}{13}$ .*

*Proof.* We assume that the theorem is false, and let  $G$  be a minimal counterexample to the theorem; by “minimal” we mean having the fewest vertices and, subject to that, the fewest non-triangular faces (thus,  $G$  is a triangulation). We will use discharging with initial charge  $\text{ch}(v) = d(v) - 6$ . We use the following five discharging rules to guarantee that each vertex finishes with nonnegative charge, which yields a contradiction.

- (R1) Each 6-vertex gives  $\frac{1}{2}$  to each 5-neighbor unless either they share a common 6-neighbor and no common 5-neighbor or else the 5-neighbor receives charge from at least four vertices; in either of these cases, the 6-vertex gives the 5-neighbor  $\frac{1}{4}$ .
- (R2) Each  $8^+$ -vertex  $v$  gives  $\frac{1}{4} + \frac{h_w}{8}$  to each  $6^-$ -neighbor  $w$  where  $h_w$  is the number of  $7^+$ -vertices in  $N(v) \cap N(w)$ .
- (R3) Each 7-vertex gives  $\frac{1}{2}$  to each isolated 5-neighbor; gives 0 to each crowded 5-neighbor; gives  $\frac{1}{4}$  to each other 5-neighbor; and gives  $\frac{1}{4}$  to each 6-neighbor unless neither the 7-vertex nor the 6-vertex has a 5-neighbor.
- (R4) After applying (R1)–(R3), each 5-vertex with positive charge splits it equally among its 6-neighbors that gave it  $\frac{1}{2}$ .
- (R5) After applying (R1)–(R4), each 6-vertex with positive charge splits it equally among its 6-neighbors with negative charge.

Now we show that after applying these five discharging rules, each vertex  $v$  finishes with nonnegative charge, i.e.,  $\text{ch}^*(v) \geq 0$ . (It is worth noting that if some vertex  $v$  has nonnegative charge after applying only (R1)–(R3), then  $v$  also has nonnegative charge after applying (R1)–(R5), i.e.,  $\text{ch}^*(v) \geq 0$ . In fact, the analysis for most cases only needs (R1)–(R3). The final two rules are used only in Cases (iv)–(vi), near the end of the proof.) Since the sum of the initial charges is  $-12$ , this contradicts our assumption that  $G$  was a minimal counterexample. Subject to proving the needed reducibility lemmas, this contradiction completes the proof of Theorem 1.

**$\mathbf{d}(\mathbf{v}) \geq 8$ :** We will show that  $v$  gives away charge at most  $\frac{d(v)}{4}$ . To see that it does, let  $v$  first give charge  $\frac{1}{4}$  to each neighbor. Now let each  $6^-$ -neighbor  $w$  take  $\frac{1}{8}$  from each  $7^+$ -vertex in  $N(v) \cap N(w)$ . Since  $G[N(v)]$  is a cycle, each  $7^+$ -neighbor gives away at most the  $\frac{1}{4}$  it got from  $v$ . Each neighbor of  $v$  has received at least as much charge as by rule (R2) and  $v$  has given away charge  $\frac{d(v)}{4}$ . Now  $\text{ch}^*(v) \geq \text{ch}(v) - \frac{1}{4}d(v) = d(v) - 6 - \frac{1}{4}d(v) = \frac{3}{4}(d(v) - 8) \geq 0$ .

**$\mathbf{d}(\mathbf{v}) = 7$ :** Let  $u_1, \dots, u_7$  denote the neighbors of  $v$  in clockwise order. First suppose that  $v$  has an isolated 5-neighbor. By Lemma 20, the subgraph induced by the remaining  $6^-$ -neighbors must have independence number at most 1. Hence  $v$  gives away charge at most either  $\frac{1}{2} + \frac{1}{2}$  or  $\frac{1}{2} + 2(\frac{1}{4})$ ; in either case,  $\text{ch}^*(v) \geq \text{ch}(v) - 1 = 0$ . Assume instead that  $v$  has no isolated 5-neighbor. Suppose first that  $v$  has a (non-isolated) 5-neighbor. Now  $v$  has at most five total  $6^-$ -neighbors, again by Lemma 20. If  $v$  has at most four  $6^-$  neighbors, then, since each  $6^-$ -neighbor receives charge at most  $\frac{1}{4}$ , we have  $\text{ch}^*(v) \geq \text{ch}(v) - 4(\frac{1}{4}) = 0$ . By Lemma 20, if  $v$  has

exactly five  $6^-$ -neighbors, then one is a crowded 5-neighbor, which receives no charge from  $v$ . So, again,  $\text{ch}^*(v) \geq 1 - 4(\frac{1}{4}) = 0$ . Finally, suppose that  $v$  has only  $6^+$ -neighbors. By Lemma 23,  $v$  gives charge to at most four 6-neighbors, so  $\text{ch}^*(v) \geq \text{ch}(v) - 4(\frac{1}{4}) = 0$ .

**d(v) = 5:** Since  $\text{ch}(v) = -1$ , we must show that  $v$  receives total charge at least 1. Let  $u_1, \dots, u_5$  be the neighbors of  $v$ . First suppose that  $v$  has five  $6^+$ -neighbors. Now  $v$  will receive charge at least  $4(\frac{1}{4})$  unless exactly two of these neighbors are 7-vertices for which  $v$  is a crowded 5-neighbor. However, in this case the other three neighbors are all 6-neighbors, so  $\text{ch}^*(v) \geq -1 + 2(\frac{1}{4}) + \frac{1}{2} = 0$ . Now suppose that  $v$  has exactly four  $6^+$ -neighbors, say  $u_1, \dots, u_4$ . If  $v$  receives charge from each, then  $\text{ch}^*(v) \geq -1 + 4(\frac{1}{4}) = 0$ ; so suppose that  $v$  receives charge from at most three neighbors. In total,  $v$  receives charge at least  $\frac{1}{2}$  from  $u_1$  and  $u_2$ : at least  $2(\frac{1}{4})$  if  $u_1$  is not a 6-vertex and at least  $\frac{1}{2} + 0$  if  $u_1$  is a 6-vertex. Similarly,  $v$  receives at least  $\frac{1}{2}$  in total from  $u_3$  and  $u_4$ ; so,  $\text{ch}^*(v) \geq -1 + 2(\frac{1}{2}) = 0$ . Now suppose that  $v$  has exactly three  $6^+$ -neighbors, say  $u_1, u_2, u_3$ . Lemma 9 implies that  $u_1, u_2, u_3$  are consecutive neighbors of  $v$ . If  $u_1$  and  $u_3$  are both 6-vertices, then  $v$  receives charge  $\frac{1}{2}$  from each. If both are  $7^+$ -vertices, then  $v$  receives charge  $\frac{1}{4}$  from each and charge  $\frac{1}{2}$  from  $u_2$ . So assume that exactly one of  $u_1$  and  $u_3$  is a 6-vertex, say  $u_1$ . Now  $v$  receives charge  $\frac{1}{2}$  from  $u_1$  and charge  $\frac{1}{4}$  from each of  $u_2$  and  $u_3$ , for a total of  $\frac{1}{2} + 2(\frac{1}{4})$ . In every case  $\text{ch}^*(v) \geq \text{ch}(v) + 1 = 0$ .

**d(v) = 6:** Note that (R5) will never cause a 6-vertex to have negative charge. Thus, in showing that a 6-vertex has nonnegative charge, we need not consider it.

Clearly, a 6-vertex with no 5-neighbor finishes (R1)–(R3) with nonnegative charge. Suppose that  $v$  is a 6-vertex with exactly one 5-neighbor. We will show that  $v$  finishes (R1)–(R3) with charge at least  $\frac{1}{4}$ . Let  $u_1, \dots, u_6$  denote the neighbors of  $v$  and assume that  $u_1$  is the only 5-vertex. By Lemma 17, at least one of  $u_1, u_3, u_5$  is a  $7^+$ -vertex, so it gives  $v$  charge  $\frac{1}{4}$ . If one of  $u_6$  and  $u_2$  is a 6-vertex, then  $v$  gives charge only  $\frac{1}{4}$  to  $u_1$ , finishing with charge at least  $2(\frac{1}{4}) - \frac{1}{4}$ . Otherwise,  $v$  receives charge at least  $\frac{1}{4}$  from each of  $u_6$  and  $u_2$ , so finishes with charge at least  $3(\frac{1}{4}) - \frac{1}{2}$ . Similarly, if  $v$  has no 5-neighbor and at least one  $8^+$ -neighbor, then  $v$  finishes (R1)–(R3) with charge at least  $\frac{1}{4}$ .

Now suppose that  $v$  has at least two 5-neighbors. By Lemma 9, At most one of  $u_1, u_3, u_5$  can be a 5-vertex. Similarly, for  $u_2, u_4, u_6$ ; hence, assume that  $v$  has exactly two 5-neighbors. These 5-neighbors can either be “across”, say  $u_1$  and  $u_4$ , or “adjacent”, say  $u_1$  and  $u_2$ .

Suppose that  $v$  has 5-neighbors  $u_1$  and  $u_4$ . Note that all of its remaining neighbors must be  $6^+$ -vertices. At least one of  $u_1, u_3, u_5$  must be a  $7^+$ -vertex; similarly for  $u_2, u_4, u_6$ . Now we show that the total net charge that  $v$  gives to  $u_3, u_4, u_5$  is 0. Similarly, the total net charge that  $v$  gives to  $u_6, u_1, u_2$  is 0. If both  $u_3$  and  $u_5$  are  $7^+$ -vertices, then  $v$  gets  $\frac{1}{4}$  from each and gives  $\frac{1}{2}$  to  $u_4$ . Otherwise, one of  $u_3$  and  $u_5$  is a 6-vertex and the other is a  $7^+$ -vertex; now  $v$  gets  $\frac{1}{4}$  from the  $7^+$ -vertex and gives only  $\frac{1}{4}$  to  $u_4$ . The same is true for  $u_6, u_1, u_2$ . Thus,  $v$  finishes with charge 0.

Suppose instead that  $v$  has 5-neighbors  $u_1$  and  $u_2$ . By Lemmas 17 and 18 either both of  $u_3$  and  $u_5$  are  $7^+$ -vertices or one is a 6-vertex and the other an  $8^+$ -vertex. The same holds for  $u_4$  and  $u_6$ . Let  $w_1, \dots, w_6$  be the common neighbors of successive pairs of vertices in the list  $u_6, u_1, u_2, u_3, u_4, u_5, u_6$ . Note that  $w_1 \leftrightarrow w_2$ , since  $u_1$  is a 5-vertex and  $\{v, u_6, w_1, w_2, u_2\} \subseteq N(u_1)$ . Similarly,  $w_2 \leftrightarrow w_3$ . (See Figure 3.) By Lemma 9, since  $G$  has no separating 3-cycle,  $w_1$  and  $w_3$  are  $6^+$ -vertices. Consider the possible degrees for  $u_3, u_4, u_5, u_6$ . Up to symmetry, they are (i)  $7^+, 7^+, 7^+, 7^+$ , (ii)  $7^+, 8^+, 7^+, 6$ , (iii)  $8^+, 7^+, 6, 7^+$ , (iv)  $8^+, 6, 6, 8^+$ , (v)  $6, 6, 8^+, 8^+$ , and (vi)  $6, 8^+, 8^+, 6$ .

In **Case (i)**,  $v$  receives charge at least  $4(\frac{1}{4})$ , so  $\text{ch}^*(v) \geq 0$ . In **Case (ii)**,  $v$  receives charge

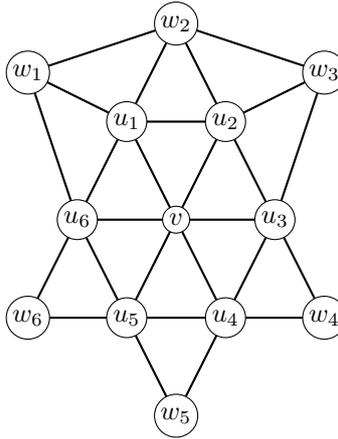


Figure 3: The closed neighborhood of  $v$  and some nearby vertices.

at least  $\frac{1}{4} + (\frac{1}{4} + \frac{1}{8} + \frac{1}{8}) + \frac{1}{4}$ , so  $\text{ch}^*(v) \geq 1 - (\frac{1}{2} + \frac{1}{4}) \geq 0$ . In **Case (iii)**,  $v$  receives charge at least  $(\frac{1}{4} + \frac{1}{8}) + \frac{1}{4} + \frac{1}{4} = \frac{7}{8}$ . Recall that  $w_3$  is a  $6^+$ -vertex, by Lemma 9. If  $w_2$  is a  $6^+$ -vertex, then  $v$  gives only  $\frac{1}{4}$  to  $u_2$ , so  $\text{ch}^*(v) \geq \frac{7}{8} - (\frac{1}{4} + \frac{1}{2}) = 0$ . So suppose that  $w_2$  is a 5-vertex. Now in each case  $v$  gets charge at least  $\frac{1}{8}$  back from  $u_2$ , via (R4). If  $w_3$  is a 6-vertex, then  $u_2$  receives charge  $2(\frac{1}{2}) + \frac{1}{4}$  and sends back  $\frac{1}{8}$  to each of  $v$  and  $w_3$ . Otherwise,  $w_3$  is a  $7^+$ -vertex, so  $u_3$  sends  $u_2$  charge at least  $\frac{3}{8}$ , and  $v$  gets back at least  $\frac{1}{8}$ . Thus, in each instance of Case (iii), we have  $\text{ch}^*(v) \geq 0$ . So we are in Cases (iv), (v), or (vi).

**Case (iv):  $8^+, 6, 6, 8^+$ .** If  $w_2$  is a  $6^+$ -vertex, then both  $u_1$  and  $u_2$  are sent charge by four vertices and hence  $v$  gives away at most  $2(\frac{1}{4})$ . Since  $v$  gets at least  $\frac{1}{4}$  from each of  $u_3$  and  $u_6$ , we have  $\text{ch}^*(v) \geq 2(\frac{1}{4}) - 2(\frac{1}{4}) = 0$ . Hence, we assume that  $w_2$  is a 5-vertex.

Now if  $w_1$  is a 6-vertex, then  $u_1$  receives charge  $\frac{5}{4}$ , so gives back  $\frac{1}{8}$  to  $v$ . If instead  $w_1$  is a  $7^+$ -vertex, then  $u_1$  receives charge at least  $\frac{3}{4}$  from  $v$  and  $w_1$  together and then charge at least  $\frac{1}{4} + \frac{1}{8}$  from  $u_6$  for a total of  $\frac{9}{8}$ . Since  $u_1$  has only one 6-neighbor, it gives the extra  $\frac{1}{8}$  back to  $v$  by (R4). The same holds for  $u_2$ , so  $v$  gets  $\frac{1}{8}$  back from each of  $u_1$  and  $u_2$ . So, the total charge that  $v$  gets from  $u_6, u_1, u_2, u_3$  is at least  $\frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \frac{3}{4}$ .

Suppose that  $u_4$  has at least two 5-neighbors. Now one of them, call it  $x$ , is a common neighbor with either  $u_3$  or  $u_5$ , so we can apply Lemma 19 to  $\{v, w_2, x\}$  (again  $x \not\leftrightarrow w_2$ , since  $w_2$  has two other 5-neighbors;  $x$  cannot be identified with one of these other 5-neighbors, since  $G$  has no separating 3-cycle). Similarly,  $u_5$  has at most one 5-neighbor. Hence, by our argument above, both  $u_4$  and  $u_5$  finish (R1)–(R3) with charge at least  $\frac{1}{4}$ . Now we show that  $u_4$  has at most three 6-neighbors; similarly for  $u_5$ .

Suppose that  $u_4$  has at least four 6-neighbors. Define  $y$  by  $N(u_4) = \{v, u_3, w_3, y, w_4, u_5\}$ . Recall that  $w_2 \leftrightarrow w_1$  and  $w_2 \leftrightarrow w_3$ , as noted before Case (i). If  $y$  is a 6-vertex, then we can apply Lemma 19 to  $\{u_5, y, u_1\}$ . (We cannot have  $y = w_1$ , since letting  $J = \{u_2, u_5, w_1\}$  gives  $|J| = 3$  and  $|N(J)| \leq 6 + 6 + 5 - 1 - 2 - 3 = 11$ , which contradicts Lemma 6.) So instead, both  $w_4$  and  $w_5$  must be 6-vertices. We can apply Lemma 19 to  $\{v, w_2, w_5\}$  unless  $w_2 \leftrightarrow w_5$ , so assume this. Also, we can apply Lemma 19 to  $\{v, w_2, w_4\}$  unless  $w_2 \leftrightarrow w_4$ ; so assume this. Hence,  $N(w_2) \supseteq \{u_1, u_2, w_1, w_3, w_4, w_5\}$ , which is a contradiction since  $d(w_2) = 5$ .

Thus, we conclude that  $u_4$  has at most two 6-neighbors other than  $u_5$ , so at most two 6-neighbors that finish (R1)–(R3) with negative charge. An analogous argument holds for  $u_5$ . Hence  $v$  gets at least  $\frac{1}{8}$  from each of  $u_4$  and  $u_5$  via (R5), so  $\text{ch}^*(v) \geq 0 - 2(\frac{1}{2}) + \frac{3}{4} + 2(\frac{1}{8}) = 0$ .

**Case (v): 6, 6, 8<sup>+</sup>, 8<sup>+</sup>.** Note that  $v$  receives charge at least  $2(\frac{3}{8}) = \frac{3}{4}$  from  $u_5$  and  $u_6$ . If  $w_2$  is a 6<sup>+</sup>-vertex, then  $u_1$  receives charge from four neighbors, so  $v$  gives away charge at most  $\frac{1}{4} + \frac{1}{2}$ . Thus  $\text{ch}^*(v) \geq 0$ . So assume  $w_2$  is a 5-vertex. First, we show that  $v$  gets back at least  $\frac{1}{8}$  from  $u_1$ . If  $d(w_1) = 6$ , then  $u_1$  gets charge  $\frac{1}{2} + \frac{1}{4} + \frac{1}{2} = \frac{5}{4}$ , so returns charge  $\frac{1}{8}$  to each of  $v$  and  $w_1$ . Otherwise  $w_1$  is a 7<sup>+</sup>-vertex, so  $u_6$  sends charge  $\frac{3}{8}$  to  $u_1$ , and  $u_1$  returns at least  $\frac{1}{8}$  to  $v$ . Thus, the total charge that  $v$  gets from  $u_5$ ,  $u_6$ , and  $u_1$  is at least  $2(\frac{3}{8}) + \frac{1}{8} = \frac{7}{8}$ .

If  $w_3$  is a 6-vertex, then  $v$  gets back charge  $\frac{1}{8}$  from  $u_2$ , via (R4), so  $\text{ch}^*(v) = 0 - 2(\frac{1}{2}) + \frac{7}{8} + \frac{1}{8} = 0$ . Instead, assume  $w_3$  is a 7<sup>+</sup>-vertex. Now we show that  $v$  gets charge at least  $\frac{1}{8}$  from  $u_3$  by (R5). Let  $y$  be the neighbor of  $u_3$  other than  $v, u_2, w_3, w_4, u_4$ . Applying Lemma 18 to  $\{u_2, u_4, y\}$ , shows that  $y$  is an 8<sup>+</sup>-vertex. If  $w_4$  is a 5-vertex, then we apply Lemma 19 to  $\{v, w_2, w_4\}$  to get a contradiction ( $w_4$  cannot be adjacent to  $w_2$ , since  $w_2$  already has two other 5-neighbors, and  $w_4$  cannot be identified with  $u_1$  or  $w_2$ , since  $G$  has no separating 3-cycles). Hence  $w_4$  is a 6<sup>+</sup>-vertex. So  $u_3$  receives charge at least  $\frac{1}{4}$  from  $w_3$  and at least  $\frac{1}{4} + \frac{1}{8}$  from  $y$ . After  $u_3$  gives charge  $\frac{1}{4}$  to  $u_2$ , it has charge at least  $\frac{3}{8}$ . So, by (R5),  $u_3$  gives each of its at most three 6-neighbors (including  $v$ ) charge at least  $\frac{1}{3}(\frac{3}{8}) = \frac{1}{8}$ . Thus,  $\text{ch}^*(v) \geq -1 + \frac{7}{8} + \frac{1}{8} = 0$ .

**Case (vi): 6, 8<sup>+</sup>, 8<sup>+</sup>, 6.** First suppose that  $w_2$  is a 6<sup>+</sup>-vertex. Note that  $v$  gets charge at least  $2(\frac{3}{8})$  from  $u_4$  and  $u_5$ , so it suffices to show that  $v$  gives net charge at most  $\frac{3}{8}$  to each of  $u_1$  and  $u_2$ . We consider  $u_1$ ; the case for  $u_2$  is symmetric. If  $w_1$  gives charge to  $u_1$ , then  $u_1$  receives charge from four neighbors, so it gets charge only  $\frac{1}{4}$  from  $v$ . Recall that  $w_1$  must be a 6<sup>+</sup>-vertex, as noted before Case (i). Thus  $w_1$  fails to give charge to  $u_1$  only if  $u_1$  is a crowded 5-neighbor of  $w_1$ ; suppose this is the case. So  $w_1$  is a 7-vertex and  $w_2$  is a 6-vertex. Now  $u_1$  gets charge  $\frac{1}{2} + \frac{1}{4} + \frac{1}{2} = \frac{5}{4}$ , so  $u_1$  returns charge  $\frac{1}{8}$  to each of  $w_2$  and  $v$ , via (R4), as desired. By symmetry,  $u_2$  also returns  $\frac{1}{8}$  to  $v$ . Thus  $\text{ch}^*(v) \geq 0 - 2(\frac{1}{2}) + 2(\frac{3}{8}) + 2(\frac{1}{8}) = 0$ . So instead, assume that  $w_2$  is a 5-vertex.

Now we show that  $u_2$  returns  $\frac{1}{8}$  to  $v$  via (R4). By symmetry the same is true of  $u_1$ . If  $w_3$  is a 6-vertex, then  $v$  gets back  $\frac{1}{8}$  from  $u_2$ , since  $u_2$  receives  $\frac{1}{2} + \frac{1}{4} + \frac{1}{2}$  and returns  $\frac{1}{8}$  to each of  $w_3$  and  $v$ . So assume, that  $w_3$  is a 7<sup>+</sup>-vertex. If  $w_4 \leftrightarrow w_2$ , then we apply Lemma 8 to  $\{w_2, u_3\}$ ; so  $w_4 \not\leftrightarrow w_2$ . If  $w_4$  is a 6<sup>-</sup>-vertex, then we apply Lemma 19 to  $\{v, w_2, w_4\}$  to get a contradiction (as above,  $w_4$  cannot be identified with  $u_1$  or  $w_2$ , since  $G$  has no separating 3-cycle). Thus,  $w_4$  is a 7<sup>+</sup>-vertex. So  $u_3$  has at least three 7<sup>+</sup>-neighbors and at most two 6-neighbors. Thus, after  $u_3$  gives charge  $\frac{1}{4}$  to  $u_2$ , by (R5) it gives charge  $\frac{1}{2}(\frac{1}{2}) = \frac{1}{4}$  to  $v$ . So in each case,  $u_3$  gives at least  $\frac{1}{8}$  to  $v$  via (R5). Since the same is true of  $u_6$ , we have  $\text{ch}^*(v) \geq 0 - 2(\frac{1}{2}) + 2(\frac{3}{8}) + 2(\frac{1}{8}) = 0$ .  $\square$

## 4 Reducibility

It is quite useful to know that a minimal counterexample has no separating 3-cycle; we prove this in Lemma 2. When proving coloring results, such a lemma is nearly trivial. However, for independence results, it requires much more work. Albertson proved an analogous lemma when showing that planar graphs have independence ratio at least  $\frac{2}{9}$ . Our proof generalizes his to the broader context of showing that all minor-closed graphs have independence ratio at least  $c$  for some rational  $c$ . We will apply this lemma to planar graphs and will let  $c = \frac{3}{13}$ .

**Lemma 2.** *Let  $c > 0$  be rational. Let  $\mathcal{G}$  be a minor-closed family of graphs. If  $G$  is a minimal counterexample to the statement that every  $n$ -vertex graph in  $\mathcal{G}$  has an independent set of size at least  $cn$ , then  $G$  has no separating 3-cycle.*

*Proof.* Suppose to the contrary that  $G$  has a separating 3-cycle  $X$ . Let  $A_1$  and  $A_2$  be induced subgraphs of  $G$  with  $V(A_1) \cap V(A_2) = X$  and  $A_1 \cup A_2 = G$ .

Our plan is to find big independent sets in two smaller graphs in  $\mathcal{G}$  (by minimality) and piece those independent sets together to get an independent set in  $G$  of size at least  $c|G|$  (for brevity, we write  $|G|$  for  $|V(G)|$ ). More precisely, we consider independent sets in each  $A_i$ , either with  $X$  deleted, or with some pair of vertices in  $X$  identified. In Claims 1–3, we prove lower bounds on  $\alpha(G)$  in terms of  $|A_1|$  and  $|A_2|$ . In Claim 4, we examine  $|A_1|$  and  $|A_2|$  modulo  $b$ , where  $c = \frac{a}{b}$  in lowest terms. In each case, we show that one of the independent sets constructed in Claims 1–3 has size at least  $c|G|$ . Our proof relies heavily on the fact that  $\alpha(H)$  is an integer (for every graph  $H$ ), which often allows us to gain slightly over  $c|H|$ .

**Claim 1.**  $\alpha(G) \geq \lceil c(|A_1| - 3) \rceil + \lceil c(|A_2| - 3) \rceil$ .

The union of the independent sets obtained by applying minimality of  $G$  to  $A_1 \setminus X$  and  $A_2 \setminus X$  is independent in  $G$ .

**Claim 2.**  $\alpha(G) \geq \lceil c(|A_i| - 2) \rceil + \lceil c|A_j| \rceil - 1$  whenever  $\{i, j\} = \{1, 2\}$ .

For concreteness, let  $i = 1$  and  $j = 2$ ; the other case is analogous. Apply minimality to  $A_2$  to get an independent set  $I_2$  in  $A_2$  with  $|I_2| \geq \lceil c|A_2| \rceil$ . Form  $A'_1$  from  $A_1$  by contracting  $X$  to a single vertex  $u$ . Apply minimality to  $A'_1$  to get an independent set  $I_1$  in  $A'_1$  with  $|I_1| \geq \lceil c(|A_1| - 2) \rceil$ . If  $u \in I_1$ , then  $I_1 \cup I_2 \setminus \{u\}$  is independent in  $G$  and has the desired size. Otherwise,  $I_1 \cup I_2 \setminus X$  is an independent set of the desired size in  $G$ .

**Claim 3.**  $\alpha(G) \geq \lceil c(|A_1| - 1) \rceil + \lceil c(|A_2| - 1) \rceil - 1$ .

Let  $X = \{x_1, x_2, x_3\}$ . For each  $k \in \{1, 2\}$  and  $t \in \{2, 3\}$ , form  $A_{k,t}$  from  $A_k$  by contracting  $x_1x_t$  to a vertex  $x_{k,t}$ . Applying minimality to  $A_{k,t}$  gives an independent set  $I_{k,t}$  in  $A_{k,t}$  with  $|I_{k,t}| \geq \lceil c(|A_k| - 1) \rceil$ .

If at most one of  $I_{1,t}$  and  $I_{2,t}$  contains a vertex of  $X$  (or a contraction of two vertices in  $X$ ), then to get a big independent set, we take their union, discarding this at most one vertex. Formally, if  $\{x_{k,t}, x_{5-t}\} \cap I_{k,t} = \emptyset$ , then  $(I_{1,t} \cup I_{2,t}) \setminus X$  is an independent set in  $G$  of the desired size. So assume that each of  $I_{1,t}$  and  $I_{2,t}$  contains a vertex (or a contraction of an edge) of  $X$ .

Now we look for a vertex  $x_\ell$  of  $X$  such that each of  $I_{1,t}$  and  $I_{2,t}$  contains  $x_\ell$  or a contraction of  $x_\ell$ . Formally, if  $x_{5-t} \in I_{1,t} \cap I_{2,t}$ , then  $(I_{1,t} \cup I_{2,t}) \setminus X$  is an independent set in  $G$  of the desired size. Similarly, if  $x_{1,t} \in I_{1,t}$  and  $x_{2,t} \in I_{2,t}$ , then  $(I_{1,t} \cup I_{2,t} \cup \{x_1\}) \setminus \{x_{1,t}, x_{2,t}\}$  is an independent set in  $G$  of the desired size.

So, by symmetry, we may assume that  $x_{1,2} \in I_{1,2}$  and  $x_3 \in I_{2,2}$ . Also, either  $x_{1,3} \in I_{1,3}$  or  $x_{2,3} \in I_{2,3}$ . If  $x_{1,3} \in I_{1,3}$ , then  $(I_{2,2} \cup I_{1,3}) \setminus \{x_{1,3}\}$  is an independent set in  $G$  of the desired size. Otherwise,  $x_{2,3} \in I_{2,3}$  and  $(I_{1,2} \cup I_{2,3} \cup \{x_1\}) \setminus \{x_{1,2}, x_{2,3}\}$  is an independent set in  $G$  of the desired size.

**Claim 4.** *The lemma holds.*

Let  $a$  and  $b$  be positive integers such that  $c = \frac{a}{b}$  and  $\gcd(a, b) = 1$ . For each  $i \in \{1, 2\}$ , let  $N_i = |A_i| - 3$  and for each  $j \in \{0, 1, 2, 3\}$ , choose  $k_i^j$  such that  $1 \leq k_i^j \leq b$  and  $k_i^j \equiv a(N_i + j) \pmod{b}$ . In other words,  $\lceil c(N_i + j) \rceil = \frac{a}{b}(N_i + j) + \frac{b - k_i^j}{b}$ . Intuitively, if there exist  $i$  and  $j$  such that  $k_i^j$  is small compared to  $b$ , then we improve our lower bound on the independence number (in some smaller graph) by the fact that the independence number is always an integer. In the present claim, we show that if some  $k_i^j$  is small, then  $G$  has an independent set of the desired size. In contrast, if all  $k_i^j$  are big, then we get a contradiction.

By symmetry, we may assume that  $k_1^0 \leq k_2^0$ .

**Subclaim 4a.**  $k_1^0 + k_2^0 \geq 2b + 1 - 3a$  and  $k_1^1 + k_2^3 \geq b + a + 1$  and  $k_1^3 + k_2^1 \geq b + a + 1$  and

$$k_1^2 + k_2^2 \geq b + a + 1.$$

If any independent set constructed in Claims 1–3 has size at least  $c|G|$ , then we are done. So we assume not; more precisely, we assume that each of these independent sets has size at most  $\frac{a|G|-1}{b}$ . Each of the four desired bounds follow from simplifying the inequalities in Claims 1–3. Note that  $|G| = N_1 + N_2 + 3$ .

By Claim 1, we have  $\alpha(G) \geq \lceil c(|A_1| - 3) \rceil + \lceil c(|A_2| - 3) \rceil = \frac{a}{b}(N_1 + N_2) + \frac{b-k_1^0}{b} + \frac{b-k_2^0}{b} = \frac{a}{b}|G| + \frac{2b-3a-k_1^0-k_2^0}{b}$ . Hence  $k_1^0 + k_2^0 \geq 2b + 1 - 3a$ .

By Claim 2, we have  $\alpha(G) \geq \lceil c(|A_1| - 2) \rceil + \lceil c|A_2| \rceil - 1 = \frac{a}{b}(N_1 + 1 + N_2 + 3) + \frac{b-k_1^1}{b} + \frac{b-k_2^3}{b} - 1 = \frac{a}{b}|G| + \frac{2b+a-k_1^1-k_2^3}{b} - 1$ . Hence  $k_1^1 + k_2^3 \geq b + a + 1$ . Similarly,  $k_1^3 + k_2^1 \geq b + a + 1$ .

By Claim 3, we have  $\alpha(G) \geq \lceil c(|A_1| - 1) \rceil + \lceil c(|A_2| - 1) \rceil - 1 \geq \frac{a}{b}(N_1 + 2 + N_2 + 2) + \frac{b-k_1^2}{b} + \frac{b-k_2^2}{b} - 1 = \frac{a}{b}|G| + \frac{2b+a-k_1^2-k_2^2}{b} - 1$ . Hence  $k_1^2 + k_2^2 \geq b + a + 1$ .

Now to get a contradiction, it suffices to show that  $k_i^j \leq a$  for some  $i \in \{1, 2\}$  and some  $j \in \{1, 2, 3\}$ ; since  $k_i^j \leq b$  for all  $i$  and  $j$ , this will contradict one of the equalities above.

**Subclaim 4b.** *Either  $k_2^1 \leq a$  or  $k_2^2 \leq a$ . In each case we get a contradiction, so the claim is true, and the lemma holds.*

By Subclaim 4a, we have  $k_1^0 + k_2^0 \geq 2b + 1 - 3a$ . By symmetry, we assumed  $k_2^0 \geq k_1^0$ , so we have  $k_2^0 \geq \frac{2b+1-3a}{2}$ . Since,  $k_2^2 \equiv k_2^0 + 2a \pmod{b}$  and  $\frac{2b+1-3a}{2} + 2a > b$ , we have  $k_2^2 \leq k_2^0 + 2a - b$ . Now we consider two cases, depending on whether  $k_2^0 \leq b - a$  or  $k_2^0 \geq b - a + 1$ . If  $k_2^0 \leq b - a$ , then  $k_2^2 \leq k_2^0 + 2a - b \leq (b - a) + 2a - b = a$ , a contradiction. Suppose instead that  $k_2^0 \geq b - a + 1$ . Now  $k_2^1 \equiv k_2^0 + a \pmod{b}$ . Since  $k_2^0 \geq b - a + 1$ , we see that  $k_2^0 + a \geq b + 1$ , so  $k_2^1 \leq k_2^0 + a - b \leq a$ , a contradiction.  $\square$

Now we turn to proving a series of lemmas showing that  $G$  cannot have too many  $6^-$ -vertices near each other. Many of these lemmas will rely on applications of the following result, which we think may be of independent interest. The idea for the proof is to find big independent sets for two smaller graphs, and piece them together to get a big independent set in  $G$ .

For  $S \subseteq V(G)$ , let the *interior* of  $S$  be  $\mathcal{I}(S) = \{x \in S \mid N(x) \subseteq S\}$ . For vertex sets  $V_1, V_2 \subseteq V(G)$  we write  $V_1 \leftrightarrow V_2$  if there exists an edge  $v_1v_2 \in E(G)$  with  $v_1 \in V_1$  and  $v_2 \in V_2$ ; otherwise, we write  $V_i \not\leftrightarrow V_j$ .

**Lemma 3.** *Let  $\mathcal{G}$  be a minor-closed family of graphs. Let  $G$  be a minimal counterexample to the statement that every  $n$ -vertex graph in  $\mathcal{G}$  has an independent set of size at least  $cn$  (for some fixed  $c > 0$ ). Let  $S_1, \dots, S_t$  be pairwise disjoint subsets of a nonempty set  $S \subseteq V(G)$  such that  $t < |S|$  and  $G[S_i]$  is connected for all  $i \in \{1, \dots, t\}$ . Now there exists  $X \subseteq \{1, \dots, t\}$  such that  $S_i \not\leftrightarrow S_j$  for all distinct  $i, j \in X$  and  $\alpha(G[\mathcal{I}(S) \cup \bigcup_{i \in X} S_i]) < |X| + \lceil c(|S| - t) \rceil$ .*

*Proof.* Suppose to the contrary that  $\alpha(G[\mathcal{I}(S) \cup \bigcup_{i \in X} S_i]) \geq |X| + \lceil c(|S| - t) \rceil$  for all  $X \subseteq \{1, \dots, t\}$  such that  $S_i \not\leftrightarrow S_j$  for all distinct  $i, j \in X$ . Create  $G'$  from  $G$  by contracting  $S_i$  to a single vertex  $w_i$  for each  $i \in \{1, \dots, t\}$  and removing the rest of  $S$ . (Note that we allow  $t = 0$ .) Since  $t < |S|$ , we have  $|G'| < |G|$  and hence minimality of  $G$  gives an independent set  $I$  in  $G'$  with  $|I| \geq c|G'| = c(|G| - |S| + t)$ . Let  $W = I \cap \{w_1, \dots, w_t\}$ . By assumption, we have  $\alpha(G[\mathcal{I}(S) \cup \bigcup_{w_i \in W} S_i]) \geq |W| + \lceil c(|S| - t) \rceil$ . If  $T$  is a maximum independent set in  $G[\mathcal{I}(S) \cup \bigcup_{w_i \in W} S_i]$ , then  $(I \setminus W) \cup T$  is an independent set in  $G$  of size at least  $|I| - |W| + |T| \geq c(|G| - |S| + t) - |W| + (|W| + \lceil c(|S| - t) \rceil) \geq c|G|$ , a contradiction.  $\square$

We will often apply Lemma 3 with  $S = J \cup N(J)$  for an independent set  $J$ . In this case, we always have  $J \subseteq \mathcal{I}(S)$ . We state this case explicitly in Lemma 4

**Lemma 4.** *Let  $\mathcal{G}$  be a minor-closed family of graphs. Let  $G$  be a minimal counterexample to the statement that every  $n$ -vertex graph in  $\mathcal{G}$  has an independent set of size at least  $cn$  (for some fixed  $c > 0$ ). No independent set  $J$  of  $G$  and nonnegative integer  $k$  simultaneously satisfy the following conditions.*

1.  $|J| \geq c(|N(J)| + k)$ .
2. For at least  $|J| - k$  vertices  $x \in J$ , there is an independent set  $\{u_x, v_x\}$  of size 2 in  $N(x) \setminus \bigcup_{y \in J \setminus \{x\}} N(y)$ .

*Proof.* Suppose the lemma is false. Let  $S = J \cup N(J)$  and  $t = |J| - k$ . Pick  $x_1, \dots, x_t \in J$  satisfying condition (2). For  $i \in \{1, \dots, t\}$ , let  $S_i = \{x_i, u_{x_i}, v_{x_i}\}$ . Applying Lemma 3, we get  $X \subseteq \{1, \dots, t\}$  such that  $S_i \not\leftrightarrow S_j$  for all distinct  $i, j \in X$  and  $\alpha(G[J \cup \bigcup_{i \in X} S_i]) < |X| + \lceil c(|S| - t) \rceil$ . By (2), we have  $\alpha(G[J \cup \bigcup_{i \in X} S_i]) \geq |(J \setminus X) \cup \bigcup_{x \in X} \{u_x, v_x\}| \geq (|J| - |X|) + 2|X| = |X| + |J|$ . Hence  $|X| + \lceil c(|S| - t) \rceil > |X| + |J|$ , giving  $\lceil c(|S| - t) \rceil > |J| \geq \lceil c(|N(J)| + k) \rceil$  by (1). But  $|S| - t = (|J| + |N(J)|) - (|J| - k) = |N(J)| + k$ ; so  $\lceil c(|S| - t) \rceil = \lceil c(|N(J)| + k) \rceil$ , contradicting the previous inequality. This contradiction finishes the proof.  $\square$

As a simple example of how to apply Lemma 4, we note that it immediately implies that every planar graph  $G$  has independence ratio at least  $\frac{1}{5}$ . By Euler's theorem,  $G$  has a  $5^-$ -vertex  $v$ . If  $d(v) \leq 4$ , then let  $G' = G \setminus (v \cup N(v))$ . Let  $I'$  be an independent set in  $G'$  of size at least  $(n - 5)/5$ , and let  $I = I' \cup \{v\}$ . If instead  $d(v) = 5$ , then apply Lemma 4, with  $c = \frac{1}{5}$ ,  $J = \{v\}$ , and  $k = 0$ ; since  $K_6$  is nonplanar,  $v$  has some pair of nonadjacent neighbors. This completes the proof.

**Lemma 5.** *Let  $\mathcal{G}$  be a minor-closed family of graphs. Let  $G$  be a minimal counterexample to the statement that every  $n$ -vertex graph in  $\mathcal{G}$  has an independent set of size at least  $cn$  (for some fixed  $c > 0$ ). For any non-maximal independent set  $J$  in  $G$ , we have*

$$|N(J)| \geq \left\lfloor \frac{1-c}{c} |J| \right\rfloor + 2.$$

*Proof.* Assume the lemma is false and choose a counterexample  $J$  minimizing  $|J|$ . Suppose  $G[J \cup N(J)]$  is not connected. Now we choose a partition  $\{J_1, \dots, J_k\}$  of  $J$ , minimizing  $k$ , such that  $k \geq 2$  and  $G[J_i \cup N(J_i)]$  is connected for each  $i \in \{1, \dots, k\}$ . Applying the minimality of  $|J|$  to each  $J_i$  we conclude that  $|N(J_i)| \geq \lfloor \frac{1-c}{c} |J_i| \rfloor + 2$  for each  $i \in \{1, \dots, k\}$ . The minimality of  $k$  gives  $|N(J)| = \left| \bigcup_{i=1}^k N(J_i) \right| = \sum_{i=1}^k |N(J_i)|$ , so  $|N(J)| \geq 2k + \sum_{i=1}^k \lfloor \frac{1-c}{c} |J_i| \rfloor \geq k + \sum_{i=1}^k \frac{1-c}{c} |J_i| \geq 2 + \frac{1-c}{c} |J|$ , a contradiction. Hence,  $G[J \cup N(J)]$  is connected.

Let  $S = J \cup N(J)$ . Apply Lemma 3 with  $t = 1$  and  $S_1 = S$ . This shows that either  $|J| \leq \alpha(G[\mathcal{I}(S)]) < \lceil c(|S| - 1) \rceil$  or  $\alpha(G[S]) < 1 + \lceil c(|S| - 1) \rceil$ , since the only possibilities are  $X = \emptyset$  and  $X = \{1\}$ . By assumption  $J$  is a counterexample, so  $|N(J)| \leq \lfloor \frac{1-c}{c} |J| \rfloor + 1$ , which implies that  $|S| = |J| + |N(J)| \leq |J| + \lfloor \frac{1-c}{c} |J| \rfloor + 1 = \left\lfloor \frac{|J|}{c} \right\rfloor + 1$ . Now  $\lceil c(|S| - 1) \rceil \leq \left\lceil c \left( \left\lfloor \frac{|J|}{c} \right\rfloor + 1 - 1 \right) \right\rceil = \left\lceil c \left\lfloor \frac{|J|}{c} \right\rfloor \right\rceil \leq \lceil |J| \rceil = |J|$ . Hence, we cannot have  $X = \emptyset$  in Lemma 3.

Instead, we must have  $X = \{1\}$ , which implies that  $\alpha(G[S]) < 1 + \lceil c(|S| - 1) \rceil$ . Since  $J$  is non-maximal, we have  $S \neq V(G)$ , so we may apply minimality of  $G$  to  $G[S]$  to conclude that  $\alpha(G[S]) \geq \lceil c|S| \rceil$ . Combining this inequality with the previous one, we have  $\lceil c|S| \rceil = \lceil c(|S| - 1) \rceil$ . Now the upper bound on  $\lceil c(|S| - 1) \rceil$  from the previous paragraph gives  $\lceil c|S| \rceil = \lceil c(|S| - 1) \rceil \leq |J|$ . Finally, applying Lemma 3 with  $t = 0$  (simply deleting  $J \cup N(J)$ ) shows that  $|J| < \lceil c(|S|) \rceil$ . These two final inequalities contradict each other, which finishes the proof.  $\square$

Lemmas 2–5 hold in a more general setting than just  $c = \frac{3}{13}$ , as we showed. In the rest of this section, we consider only a planar graph  $G$  that is minimal among those with independence ratio less than  $\frac{3}{13}$ . To remind the reader of this, we often call it a *minimal*  $G$ . Applying Lemma 5 with  $c = \frac{3}{13}$  gives the following corollary.

**Lemma 6.** *For any non-maximal independent set  $J$  in a minimal  $G$ , we have*

$$|N(J)| \geq \left\lfloor \frac{10}{3}|J| \right\rfloor + 2.$$

*In particular, if  $|J| = 1$ , then  $|N(J)| \geq 5$ ; if  $|J| = 2$ , then  $|N(J)| \geq 8$ ; and if  $|J| = 3$ , then  $|N(J)| \geq 12$ .*

The case  $|J| = 1$  shows that  $G$  has minimum degree 5, and this is the best we can hope for when  $|J| = 1$ . Recall that  $G$  is a planar triangulation, since we chose it to have as few non-triangular faces as possible. As a result, we can improve the bound when  $|J| = 2$  to  $|N(J)| \geq 9$ . Similarly, in many cases we can improve the bound when  $|J| = 3$  to  $|N(J)| \geq 13$ . These improvements are the focus of the next ten lemmas. In many instances, the proofs are easy applications of Lemma 3. First, we need a few basic facts about planar graphs.

**Lemma 7.** *If  $G$  is a plane triangulation with no separating 3-cycle and  $\delta(G) = 5$ , then*

- (a) *If  $v \in V(G)$ , then  $G[N(v)]$  is a cycle; and*
- (b)  *$G$  is 4-connected with  $|V(G)| \geq 12$ ; and*
- (c) *If  $v, w \in V(G)$  are distinct, then  $G[N(v) \cap N(w)]$  is the disjoint union of copies of  $K_1$  and  $K_2$ .*

*Proof.* Plane triangulations are well-known to be 3-connected. Property (a) follows by noting that  $G \setminus \{v\}$  is 2-connected and hence each face boundary is a cycle; so  $G[N(v)]$  has a hamiltonian cycle. This cycle must be induced since  $G$  has no separating 3-cycle.

For (b), suppose that  $G$  has a separating set  $\{x, y, z\}$ . Since  $G$  has no separating 3-cycle, we assume that  $xy \notin E(G)$ . By (a),  $N(x)$  induces a cycle  $C$ . Since  $G$  is 3-connected,  $x$  must have a neighbor in each component of  $G \setminus \{x, y, z\}$ . So  $C$  has a vertex in each component of  $G \setminus \{x, y, z\}$  and hence  $C \setminus \{x, y, z\}$  is disconnected. But  $x \notin V(C)$  and since  $xy \notin E(G)$ , also  $y \notin V(C)$ . So,  $C \setminus \{z\}$  is disconnected, which is impossible. Since  $G$  is a plane triangulation and  $\delta(G) = 5$ , we have  $5|G| \leq 2|E(G)| = 6|G| - 12$ , so  $|G| \geq 12$ .

By (a) and  $\delta(G) = 5$ , it follows that no neighborhood contains  $K_3$  or  $C_4$ . If  $G[N(v) \cap N(w)]$  had an induced  $P_3$  (path on 3 vertices), then the neighborhood of the center of this  $P_3$  would contain  $K_3$  or  $C_4$ . This proves (c).  $\square$

**Lemma 8.** *Every independent set  $J$  in a minimal  $G$  with  $|J| = 2$ , satisfies  $|N(J)| \geq 9$ .*

*Proof.* By Lemma 7(b),  $|G| \geq 12$ ; so  $J$  cannot be a maximal independent set when  $|N(J)| \leq 7$ . Hence, by Lemma 6, we may assume  $|N(J)| = 8$ . Let  $J = \{x, y\}$ . If we can apply Lemma 4 with  $k = 0$ , then we are done. If we cannot, then by symmetry we may assume that there is no independent 2-set in  $N(x) \setminus N(y)$ . So  $N(x) \setminus N(y)$  is a clique. Since  $d(x) \geq 5$  and  $N(x)$  induces a cycle,  $|N(x) \setminus N(y)| \leq 2$ . Now, since  $x$  is a  $5^+$ -vertex,  $G[N(x) \cap N(y)]$  induces  $P_3$ ; this contradicts Lemma 7(c).  $\square$

A direct consequence of Lemma 8 is the following useful fact.

**Lemma 9.** *A minimal  $G$  cannot have two nonadjacent 5-vertices with at least two common neighbors. In particular, each vertex  $v$  in  $G$  has  $\frac{d(v)}{2}$  or more  $6^+$ -neighbors.*

*Proof.* The first statement follows immediately from Lemma 8. Now we consider the second. Let  $v$  be a vertex with  $d(v) = k$  and neighbors  $u_1, \dots, u_k$  in clockwise order. If more than  $k/2$  neighbors of  $v$  are 5-vertices, then (by Pigeonhole) there exists an integer  $i$  such that  $u_i$  and  $u_{i+2}$  are 5-vertices (subscripts are modulo  $k$ ). Now we apply Lemma 8 to  $u_i$  and  $u_{i+2}$ . Recall that  $u_i$  and  $u_{i+2}$  are nonadjacent, since  $G$  has no separating 3-cycle, as shown in Lemma 2.  $\square$

Now we consider the case when  $|J| = 3$ . Lemma 6 gives  $|N(J)| \geq 12$ . Our next few lemmas show certain conditions under which we can conclude that  $|N(J)| \geq 13$ .

**Lemma 10.** *Let  $J$  be an independent set in a minimal  $G$  with  $|J| = 3$  and  $|N(J)| \geq 12$ . Choose  $S_1, S_2 \subseteq J \cup N(J)$  such that  $S_1 \cap S_2 = \emptyset$  and both  $G[S_1]$  and  $G[S_2]$  are connected. If  $\alpha(G[S_i \cup J]) \geq 4$  for each  $i \in \{1, 2\}$ , then  $|N(J)| \geq 13$ .*

*Proof.* Suppose not and choose a counterexample minimizing  $|J \cup N(J)| - |S_1 \cup S_2|$ . Clearly  $|N(J)| = 12$ . First we show that  $S_1 \cup S_2 = J \cup N(J)$ . It suffices to show that  $G[J \cup N(J)]$  is connected, since then we can add to either  $S_1$  or  $S_2$  any vertex in  $N(S_1 \cup S_2) \setminus (S_1 \cup S_2)$ . In particular, we show that every  $x \in J$  satisfies  $(x \cup N(x)) \cap (\cup_{y \in (J \setminus \{x\})} (y \cup N(y))) \neq \emptyset$ . Suppose not. By Lemma 8, we have  $|\cup_{y \in J \setminus \{x\}} N(y)| \geq 9$ . Now  $|\cup_{y \in J} N(y)| \geq 9 + d(x) \geq 14$ , a contradiction. Now we must have  $G[J \cup N(J)]$  connected, so we can assume  $S_1 \cup S_2 = J \cup N(J)$ . Similarly, we assume  $S_1 \leftrightarrow S_2$ .

Now we apply Lemma 3 with  $S = J \cup N(J)$ ,  $t = 2$ , and  $S_1$  and  $S_2$  as above. Since  $S_1 \leftrightarrow S_2$ , we have  $|X| \leq 1$ . We cannot have  $|X| = 1$  since, by hypothesis,  $\alpha(G[S_i \cup J]) \geq 4$  for each  $i \in \{1, 2\}$ . So suppose that  $X = \emptyset$ . Now we have  $\alpha(G[J]) \geq |J| = 3 = \lceil \frac{3}{13}(|J \cup N(J)| - 2) \rceil = \lceil \frac{3}{13}(3 + 12 - 2) \rceil$ . This contradiction completes the proof.  $\square$

**Lemma 11.** *Let  $J = \{u_1, u_2, u_3\}$ . If  $J$  is an independent set in a minimal  $G$  where*

1.  $N(u_1) \setminus (N(u_2) \cup N(u_3))$  contains an independent 2-set; and
2.  $\alpha(G[J \cup N(u_2) \cup N(u_3)]) \geq 4$ ,

then  $|N(J)| \geq 13$ .

*Proof.* Since  $G$  is a planar triangulation with minimum degree 5 and at least three  $6^+$ -vertices by Lemma 9, we have  $5|G| + 3 \leq 2|E(G)| = 6|G| - 12$  and hence  $|G| \geq 15$ . Thus  $J$  cannot be a maximal independent set when  $|N(J)| \leq 11$ . So, by Lemma 6, we know that  $|N(J)| \geq 12$ . Let  $I$  be an independent set of size 2 in  $N(u_1) \setminus (N(u_2) \cup N(u_3))$ .

First, suppose  $N(u_2) \cap N(u_3) \neq \emptyset$ . We apply Lemma 10 with  $S_1 = \{u_1\} \cup I$  and  $S_2 = \{u_2, u_3\} \cup N(u_2) \cup N(u_3)$ . Clearly,  $G[S_1]$  is connected. Also,  $G[S_2]$  is connected since  $N(u_2) \cap$

$N(u_3) \neq \emptyset$ , by assumption. The set  $I \cup \{u_2, u_3\}$  shows that  $\alpha(G[S_1 \cup J]) \geq 4$  and hypothesis (2) shows that  $\alpha(G[S_2 \cup J]) \geq 4$ . So the hypotheses of Lemma 10 are satisfied, giving  $|N(J)| \geq 13$ .

Instead, suppose  $N(u_2) \cap N(u_3) = \emptyset$ . This implies  $N(u_2) \setminus (N(u_1) \cup N(u_3)) = N(u_2) \setminus N(u_1)$ . If  $N(u_2) \setminus N(u_1)$  contains an independent 2-set as well, then applying Lemma 4 with  $k = 1$  gives  $|N(J)| \geq 13$ , as desired. Otherwise,  $|N(u_2) \setminus N(u_1)| \leq 2$ , so  $G[N(u_2) \cap N(u_1)]$  contains  $P_3$ , contradicting Lemma 7(c).  $\square$

One particular case of Lemma 11 is easy to verify in our applications, so we state it separately, as Lemma 13. First, we need the following lemma.

**Lemma 12.** *Let  $v$  be a  $7^+$ -vertex in  $G$ . If  $S \subseteq V(G)$  with  $\{v\} \cup N(v) \subseteq S$  and  $|S| \geq 10$ , then  $\alpha(G[S]) \geq 4$ .*

*Proof.* If  $d(v) \geq 8$ , then the neighbors of  $v$  induce an  $8^+$ -cycle (by Lemma 7(a)), which has independence number at least 4; so we are done. So suppose  $d(v) = 7$ . Let  $u_1, \dots, u_7$  denote the neighbors of  $v$  in clockwise order; note that  $G[N(v)]$  is a 7-cycle, again by Lemma 7(a). Pick  $w_1, w_2 \in S \setminus (\{v\} \cup N(v))$ . Let  $H_i = G[N(v) \setminus N(w_i)]$  for each  $i \in \{1, 2\}$ . If  $H_i$  contains an independent 3-set  $J$  for some  $i \in \{1, 2\}$ , then  $J \cup \{w_i\}$  is the desired independent 4-set, so we are done. Therefore, we must have  $|H_i| \leq 4$  for each  $i \in \{1, 2\}$ . So,  $|N(v) \cap N(w_i)| \geq 3$  and hence Lemma 7(c) shows that  $N(v) \cap N(w_i)$  has at least two components; therefore, so does  $H_i$ . It must have exactly two components or we get an independent 3-set in  $H_i$ . Similarly, if  $|H_i| = 4$ , then  $H_i$  has no isolated vertex. So, either  $H_i$  is  $2K_2$  or  $|H_i| \leq 3$ . Now in each case we get a subdivision of  $K_{3,3}$ ; the branch vertices of one part are  $v, w_1, w_2$  and the branch vertices of the other are three of the  $u_i$ . This contradiction finishes the proof.  $\square$

**Lemma 13.** *Let  $J = \{u_1, u_2, u_3\}$ . If  $J$  is an independent set in a minimal  $G$  where*

1.  $N(u_1) \setminus (N(u_2) \cup N(u_3))$  contains an independent 2-set; and
2.  $G[J \cup N(u_2) \cup N(u_3)]$  contains a  $7^+$ -vertex and its neighborhood,

then  $|N(J)| \geq 13$ .

*Proof.* We apply Lemma 11 using Lemma 12 to verify hypothesis (2). To do so, we let  $S = \{u_1, u_2, u_3\} \cup N(u_2) \cup N(u_3)$ , and we need that  $|\{u_1, u_2, u_3\} \cup N(u_2) \cup N(u_3)| \geq 10$ . This is immediate from Lemma 8, since  $|S| \geq |\{u_1, u_2, u_3\}| + |N(u_2) \cup N(u_3)| \geq 3 + 9 = 12$ .  $\square$

**Lemma 14.** *Let  $J$  be an independent 3-set in  $G$ . Choose  $S_1, S_2, S_3 \subseteq J \cup N(J)$  such that  $G[S_i]$  is connected and  $S_i \cap S_j = \emptyset$  for all distinct  $i, j \in \{1, 2, 3\}$ . If  $|N(J)| \leq 13$ , then either*

1.  $S_i \not\leftrightarrow S_j$  for some  $\{i, j\} \subseteq \{1, 2, 3\}$ ; or
2.  $\alpha(G[S_i \cup J]) \leq 3$  for some  $i \in \{1, 2, 3\}$ .

*Proof.* This is an immediate corollary of Lemma 3 with  $S = J \cup N(J)$  and  $t = 3$ . If  $S_i \leftrightarrow S_j$  for all  $\{i, j\} \in \{1, 2, 3\}$ , then in Lemma 3 either  $|X| = 1$  or  $|X| = 0$ . We cannot have  $|X| = 0$ , since  $\alpha(G[\mathcal{I}(S)]) \geq \alpha(G[J]) \geq |J| = 3 = \lceil \frac{3}{13}(13 + 3 - 3) \rceil$ . Hence  $|X| = 1$ , which implies (2).  $\square$

The next lemma can be viewed as a variant on the result we get by applying Lemma 4 with  $|J| = 3$  and  $k = 0$  (and  $c = \frac{3}{13}$ ). As in that case, we require that each of  $N(u_1) \setminus (N(u_2) \cup N(u_3))$  and  $N(u_2) \setminus (N(u_1) \cup N(u_3))$  contains an independent 2-set. However, here we do not require that  $N(u_3) \setminus (N(u_1) \cup N(u_2))$  contains an independent 2-set. Instead, we have hypothesis (2) below. Not surprisingly, the proof is similar to that of Lemma 4.

**Lemma 15.** Let  $J = \{u_1, u_2, u_3\}$ . If  $J$  is an independent set in a minimal  $G$  such that

1.  $N(u_i) \setminus (N(u_j) \cup N(u_3))$  contains an independent 2-set  $M_i$  for all  $\{i, j\} = \{1, 2\}$ ; and
2.  $\alpha(G[J \cup V(H)]) \geq 4$ , where  $H$  is  $u_3$ 's component in  $G[\{u_3\} \cup N(J)] \setminus (M_1 \cup M_2)$ ,

then  $|N(J)| \geq 14$ .

*Proof.* First, we show that  $u_3$  is distance two from each of  $u_1$  and  $u_2$ . Suppose not; by symmetry, assume that  $u_3$  is distance at least three from  $u_1$ . Now  $N(u_3) \setminus (N(u_1) \cup N(u_2)) = N(u_3) \setminus N(u_2)$ . By Lemma 7,  $N(u_3) \cap N(u_2)$  consists of disjoint copies of  $K_1$  and  $K_2$ . Thus, since  $d(u_3) \geq 5$ , we see that  $N(u_3) \setminus (N(u_1) \cup N(u_2))$  contains an independent 2-set. Now, if  $|N(J)| \leq 13$ , then applying Lemma 4 with  $k = 0$  gives a contradiction. Hence,  $u_3$  is distance two from each of  $u_1$  and  $u_2$ .

Choose disjoint subsets  $S_1, S_2, S_3 \subset J \cup N(J)$  where  $G[S_i]$  is connected for all  $i \in \{1, 2, 3\}$  and  $\{u_i\} \cup M_i \subseteq S_i$  for each  $i \in \{1, 2\}$  and  $u_3 \in S_3$ , first maximizing  $|S_3|$  and subject to that maximizing  $|S_1| + |S_2| + |S_3|$ . Since  $J \subseteq S_1 \cup S_2 \cup S_3$ , maximality of  $|S_1| + |S_2| + |S_3|$  gives  $S_1 \cup S_2 \cup S_3 = J \cup N(J)$ .

Now we apply Lemma 3, with  $S = S_1 \cup S_2 \cup S_3$ . To get a contradiction, we need only verify, for each possible  $X$ , that  $\alpha(G[\mathcal{I}(S) \cup \bigcup_{i \in X} S_i]) \geq |X| + \lceil \frac{3}{13}(|S| - |J|) \rceil = |X| + 3$ . Since  $S_3 \leftrightarrow S_1$  and  $S_3 \leftrightarrow S_2$ , either  $|X| \leq 1$  or else  $X = \{1, 2\}$ . In the latter case,  $M_1 \cup M_2 \cup \{u_3\}$  is the desired independent 5-set. If instead  $X = \emptyset$ , then  $J$  is the desired independent 3-set.

So we must have  $X = \{i\}$  for some  $i \in \{1, 2, 3\}$ . If  $i \in \{1, 2\}$ , then  $M_i \cup \{u_3, u_{3-i}\}$  is the desired independent set. So instead assume that  $X = \{3\}$ . But, by the maximality of  $|S_3|$ ,  $G[J \cup S_3]$  contains  $u_3$ 's component in  $G[\{u_3\} \cup N(J)] \setminus M_1 \setminus M_2$ . So by (2),  $G[J \cup S_3]$  has an independent 4-set, as desired.  $\square$

Again, one particular case of Lemma 15 is easy to verify, so we state it separately.

**Lemma 16.** Let  $J = \{u_1, u_2, u_3\}$ . If  $J$  is an independent set in a minimal  $G$  such that

1.  $N(u_i) \setminus (N(u_j) \cup N(u_3))$  contains an independent 2-set  $M_i$  for all  $\{i, j\} = \{1, 2\}$ ; and
2.  $u_3$ 's component  $H$  in  $G[\{u_3\} \cup N(J)] \setminus (M_1 \cup M_2)$  satisfies  $|J \cup V(H)| \geq 10$  and  $G[J \cup V(H)]$  contains a  $7^+$  vertex and its neighborhood,

then  $|N(J)| \geq 14$ .

*Proof.* We apply Lemma 15, using Lemma 12 to verify hypothesis (2).  $\square$

Thus far, our lemmas have not focused much on the actual planar embedding of  $G$ . At this point we transition and start analyzing the embedding, as well.

**Lemma 17.** Every minimal  $G$  has no 6-vertex  $v$  with  $6^-$ -neighbors  $u_1, u_2$ , and  $u_3$  that are pairwise nonadjacent.

*Proof.* Lemma 6, applied with  $J = \{u_1, u_2, u_3\}$ , yields  $12 \leq |N(\{u_1, u_2, u_3\})| \leq d(u_1) + d(u_2) + d(u_3) - 5$ . Hence, by symmetry, assume that the vertices are arranged as in Figure 4(a) with all vertices distinct as drawn or as in Figure 4(b) with at most one pair of vertices identified.

The first case is impossible by Lemma 4 with  $k = 1$ , using the vertices labeled 2 for  $u_2$  and those labeled 3 for  $u_3$ . When the vertices in Figure 4(b) are distinct as drawn, we apply Lemma 4

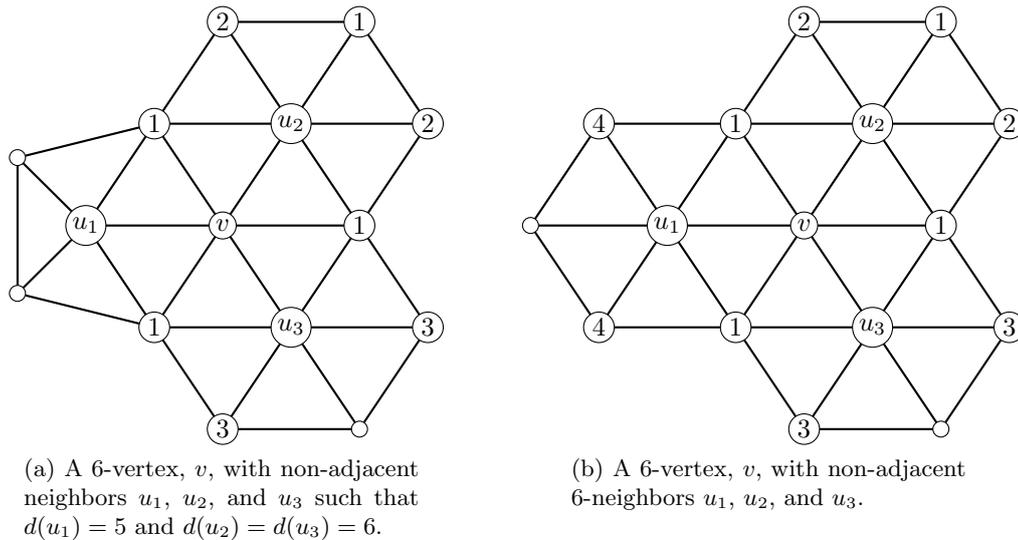


Figure 4: The two cases of Lemma 17.

with  $k = 0$ , using the vertices labeled 2 for  $u_2$ , the vertices labeled 3 for  $u_3$ , and those labeled 4 for  $u_1$ . Instead, by symmetry and the fact that  $G$  contains no separating 3-cycle, assume that the vertices labeled 2 and 3 that are drawn at distance four are identified; so  $|N(\{u_1, u_2, u_3\})| = 12$ . Now the pairs of vertices labeled 1 each have a common neighbor, so the vertices labeled 1 must be an independent set, to avoid a separating 3-cycle. Now, we apply Lemma 11, using the vertices labeled 4 for the independent 2-set. This implies that  $|N(\{u_1, u_2, u_3\})| \geq 13$ , which contradicts our conclusion above that  $|N(\{u_1, u_2, u_3\})| = 12$ .  $\square$

**Lemma 18.** *Every minimal  $G$  has no 6-vertex  $v$  with pairwise nonadjacent neighbors  $u_1$ ,  $u_2$ , and  $u_3$ , where  $d(u_1) = 5$ ,  $d(u_2) \leq 6$ , and  $d(u_3) = 7$ .*

*Proof.* Let  $J = \{u_1, u_2, u_3\}$ . By Lemma 6,  $12 \leq |N(J)| \leq 5 + 6 + 7 - 5 = 13$ , so at most one pair of vertices in Figure 5(a) are identified.

First, suppose the vertices in the figure are distinct as drawn. Suppose  $x \not\leftrightarrow y$ , as in Figure 5(b). For each  $i \in \{1, 2, 3\}$ , let  $S_i$  consist of the vertices labeled  $i$ . Now for each  $i \in \{1, 2, 3\}$ ,  $G[S_i]$  is connected. Clearly, for each  $i \in \{1, 2\}$  the vertices labeled  $I_i$  form an independent 4-set. Since  $x \not\leftrightarrow y$ , the vertices labeled  $I_3$  also form an independent 4-set. Note that  $S_1 \leftrightarrow S_3$  and  $S_2 \leftrightarrow S_3$ ; however, possibly  $S_1 \not\leftrightarrow S_2$ . If  $S_1 \leftrightarrow S_2$ , then we can apply Lemma 14 to get a contradiction. So, we assume that  $S_1 \not\leftrightarrow S_2$ . But now we have an independent 5-set consisting of  $u_1$ , the two vertices labeled  $\{1, I_1\}$  and the two vertices labeled  $\{2, I_2\}$ ; hence  $\alpha(G[S_1 \cup S_2 \cup J]) \geq 5$ . So, we can apply Lemma 3 to get a contradiction. So, instead we assume  $x \leftrightarrow y$ .

Suppose  $w \not\leftrightarrow z$ , as in Figure 5(c). For each  $i \in \{1, 2, 3\}$ , let  $S_i$  consist of the vertices labeled  $i$ . Clearly  $G[S_i]$  is connected for each  $i \in \{2, 3\}$ . Also,  $G[S_1]$  is connected because  $x \leftrightarrow y$ . Note that for each  $i \in \{1, 3\}$ , the vertices labeled  $I_i$  form an independent 4-set. Since  $x \leftrightarrow y$  and  $w \not\leftrightarrow z$ , the vertices labeled  $I_2$  also form an independent 4-set. Note that  $S_1 \leftrightarrow S_2$  and  $S_2 \leftrightarrow S_3$ ; however, possibly  $S_1 \not\leftrightarrow S_3$ . If  $S_1 \leftrightarrow S_3$ , then we apply Lemma 14 to get a contradiction. So instead we assume that  $S_1 \not\leftrightarrow S_3$ . But now we again have an independent 5-set, consisting of  $u_1$ ,

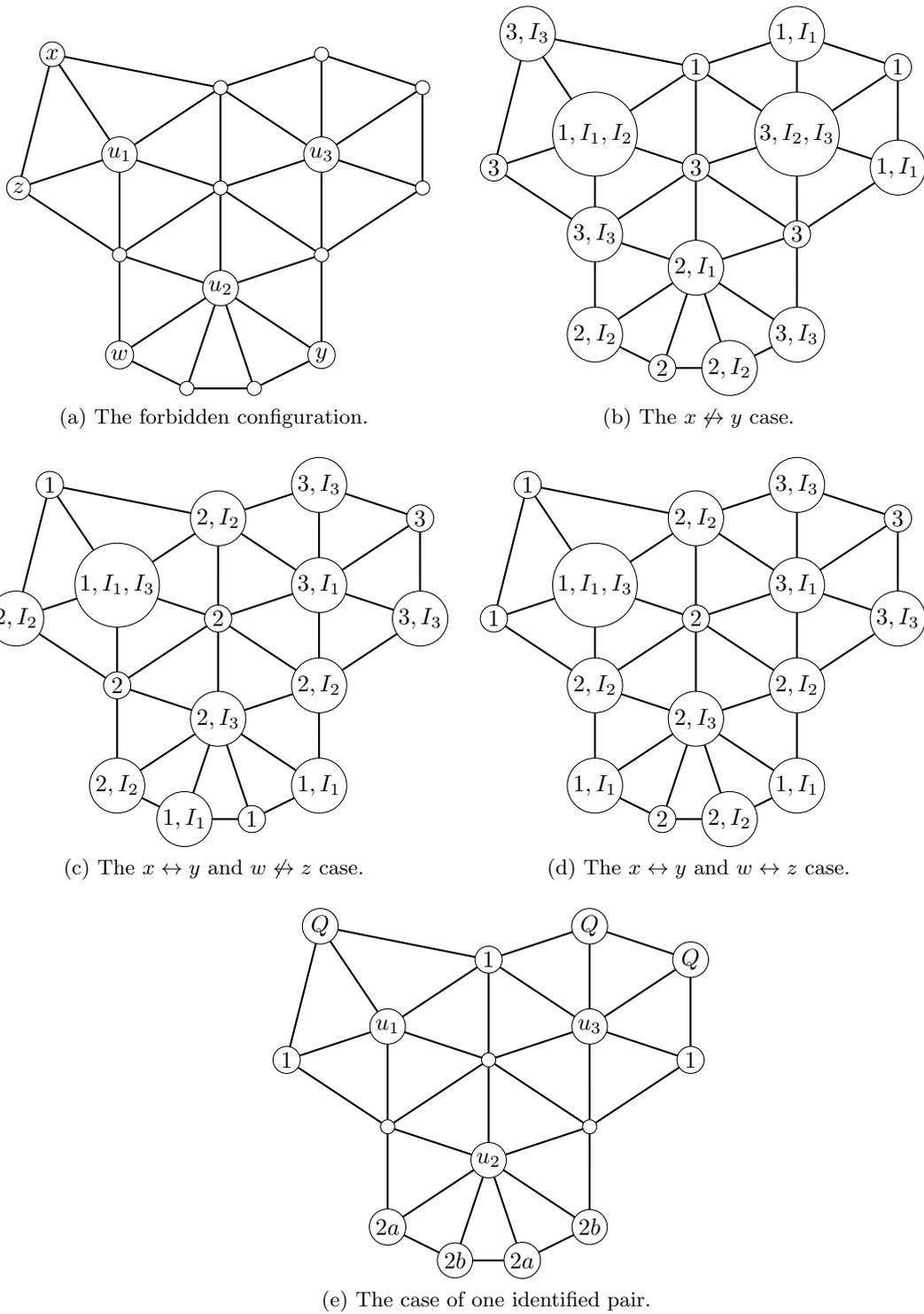


Figure 5: The case of Lemma 18.

the two vertices labeled  $\{1, I_1\}$ , and the two vertices labeled  $\{3, I_3\}$ ; hence  $\alpha(G[S_1 \cup S_3 \cup J]) \geq 5$ . So, again we apply Lemma 3 to get a contradiction. Thus, we instead assume  $w \leftrightarrow z$ .

Now consider Figure 5(d). For each  $i \in \{1, 2, 3\}$ , let  $S_i$  consist of the vertices labeled  $i$ . Note that  $G[S_i]$  is connected for each  $i \in \{2, 3\}$ . Also,  $G[S_1]$  is connected because  $x \leftrightarrow y$  and  $w \leftrightarrow z$ . Clearly, the vertices labeled  $I_i$  form an independent 4-set for each  $i \in \{1, 3\}$ . Since  $x \leftrightarrow y$ , the vertices labeled  $I_2$  also form an independent 4-set. Note that  $S_1 \leftrightarrow S_2$  and  $S_2 \leftrightarrow S_3$ ; however, possibly  $S_1 \not\leftrightarrow S_3$ . If  $S_1 \leftrightarrow S_3$ , then we apply Lemma 14 to get a contradiction. So, instead we assume that  $S_1 \not\leftrightarrow S_3$ . But now we have an independent 5-set, consisting of  $u_1$ , the two vertices labeled  $\{1, I_1\}$ , and the two vertices labeled  $\{3, I_3\}$ ; hence  $\alpha(G[S_1 \cup S_3 \cup J]) \geq 5$ . So, we apply Lemma 3 to get a contradiction.

Hence, we may assume that exactly one pair of vertices in Figure 5(a) is identified. No neighbor of  $u_1$  can be identified with a neighbor of  $u_3$ , since then  $u_1$  and  $u_3$  would have three common neighbors, violating Lemma 8. Hence, to avoid separating 3-cycles, we assume that a vertex labeled  $2a$  is identified with a vertex labeled  $Q$  (the case where a vertex labeled  $2b$  is identified with a vertex labeled  $Q$  is nearly identical, so we omit the details). But now the rightmost vertex labeled 1 and the leftmost vertex labeled 1 are on opposite sides of a separating cycle and hence nonadjacent. Therefore,  $u_2$  together with the vertices labeled 1 is an independent 4-set. So, now we apply Lemma 11 to get a contradiction, using the vertices labeled  $2b$  for the independent 2-set.  $\square$

**Lemma 19.** *Let  $u_1$  be a 6-vertex with nonadjacent vertices  $u_2$  and  $u_3$  each at distance two from  $u_1$ , where  $u_2$  is a 5-vertex and  $u_3$  is a  $6^-$ -vertex. A minimal  $G$  cannot have  $u_1$  and  $u_2$  with two common neighbors, and also  $u_1$  and  $u_3$  with two common neighbors.*

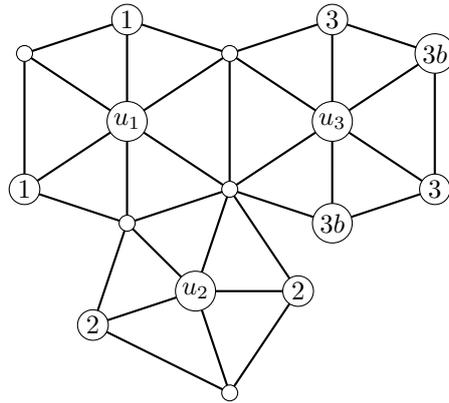
*Proof.* Figure 6 shows the possible arrangements when  $u_3$  is a 6-vertex. The case when  $u_3$  is a 5-vertex is similar, but easier. In particular, when  $u_3$  is a 5-vertex, we already know  $|N(\{u_1, u_2, u_3\})| \leq 12$ , so all vertices in the corresponding figures must be distinct as drawn. Furthermore, it now suffices to apply Lemma 4 with  $k = 1$ . We omit further details. So suppose instead that  $d(u_3) = 6$ .

First, suppose all vertices in the figures are distinct as drawn. Now Figures 6(a,c) are impossible by Lemma 4 with  $k = 0$ ; for each  $i \in \{1, 2, 3\}$ , we use the vertices labeled  $i$  as the independent 2-set for  $u_i$ . For Figure 6(b), let  $I_1$  be the vertices labeled  $u_2$  or  $1a$  and let  $I_2$  be the vertices labeled  $u_2$  or  $1b$ . To avoid a separating 3-cycle, at least one of  $I_1$  or  $I_2$  is independent. Hence Figure 6(b) is impossible by Lemma 15; for the independent 4-set, use  $I_1$  or  $I_2$  and for each  $i \in \{2, 3\}$ , use the vertices labeled  $i$  as the independent 2-set for  $u_i$ .

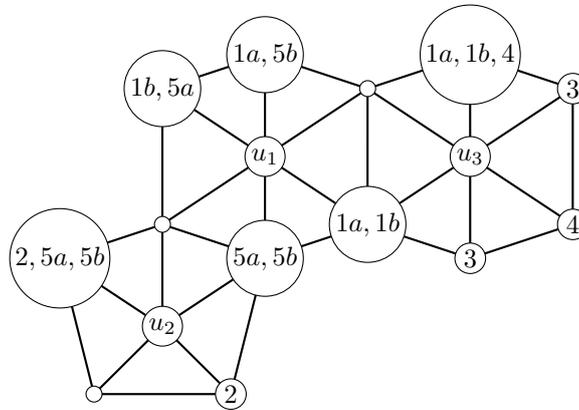
By Lemma 6,  $|N(J)| \geq 12$ , so exactly one pair of vertices is identified in one of Figures 6(a,b,c). First, consider Figures 6(a,c) simultaneously. Since  $G$  has no separating 3-cycle, the identified pair must contain a vertex labeled 3. Now we apply Lemma 4 with  $k = 1$ , using the vertices labeled  $3b$  in place of those labeled 3.

Finally, for Figure 6(b), we apply Lemma 11. For the independent 2-set we use either the vertices labeled 3 or the vertices labeled 4; at least one of these pairs contains no identified vertex. For the independent 4-set, we use either  $u_3$  and the vertices labeled  $5a$  or else  $u_3$  and the vertices labeled  $5b$ . Since  $G$  has no separating 3-cycle, at least one of these 4-sets will be independent.  $\square$

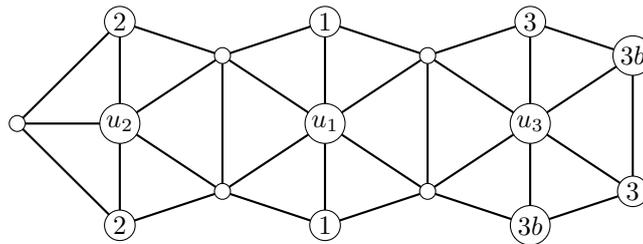
**Lemma 20.** *Every minimal  $G$  has no 7-vertex  $v$  with a 5-neighbor and two other  $6^-$ -neighbors,  $u_1$ ,  $u_2$ , and  $u_3$ , that are pairwise nonadjacent. In other words, Figures 7(a–e) are forbidden.*



(a) Here  $u_2$  and  $u_3$  have a common neighbor in  $N(u_1)$ .

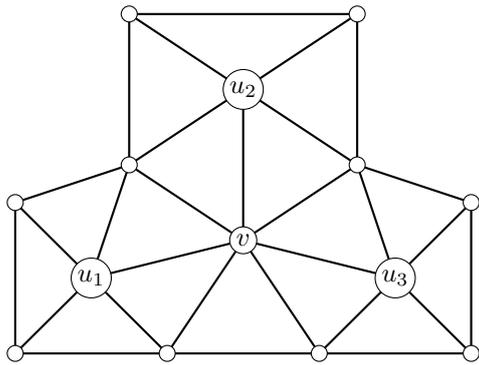


(b) Here  $u_2$  and  $u_3$  have adjacent neighbors in  $N(u_1)$ .

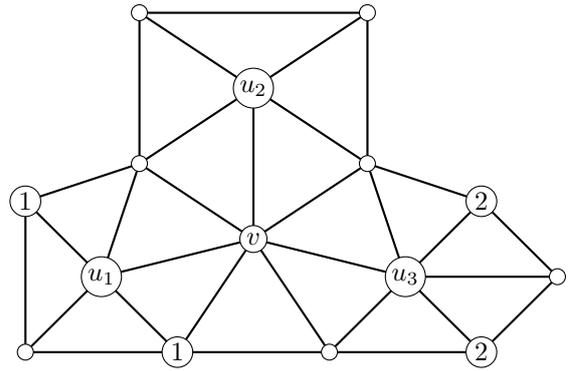


(c) Here  $u_2$  and  $u_3$  have neighbors at distance 2 in  $N(u_1)$ .

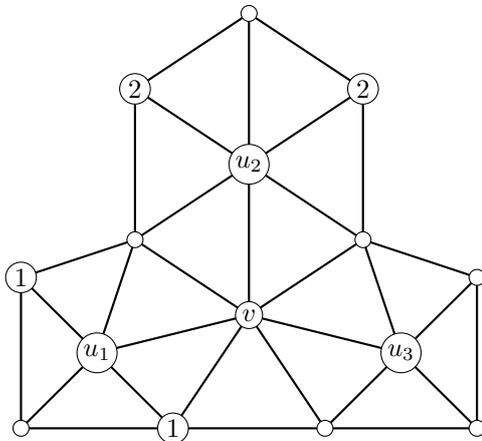
Figure 6: The cases of Lemma 19. The three possibilities for an independent 3-set  $\{u_1, u_2, u_3\}$  where  $d(u_1) = 6$ ,  $d(u_2) \leq 6$ ,  $d(u_3) = 5$ , and each of  $u_2$  and  $u_3$  has two neighbors in common with  $u_1$ .



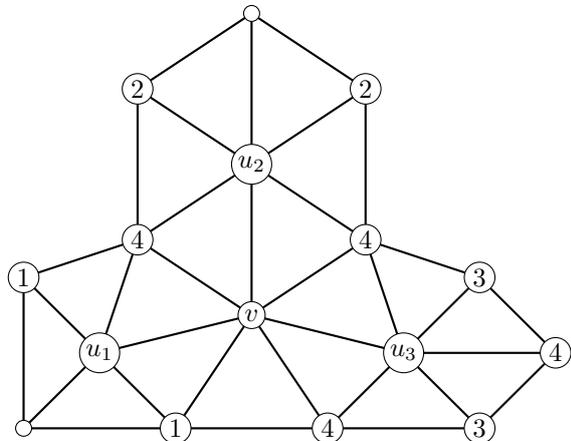
(a) A 7-vertex,  $v$ , with non-adjacent 5-neighbors,  $u_1$ ,  $u_2$ , and  $u_3$ .



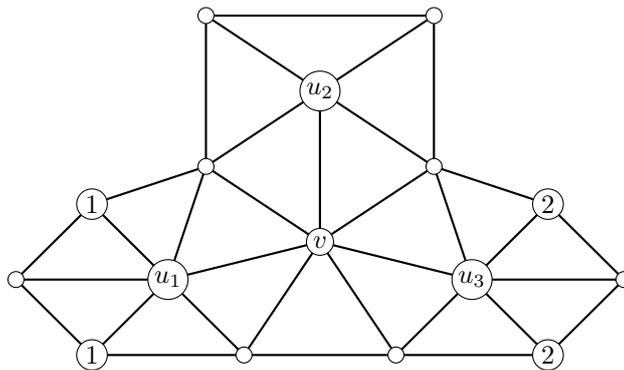
(b) A 7-vertex,  $v$ , with a 6-neighbor,  $u_3$ , and two 5-neighbors,  $u_1$  and  $u_2$ .



(c) A 7-vertex,  $v$ , with a 6-neighbor,  $u_2$ , and two 5-neighbors,  $u_1$  and  $u_3$ .

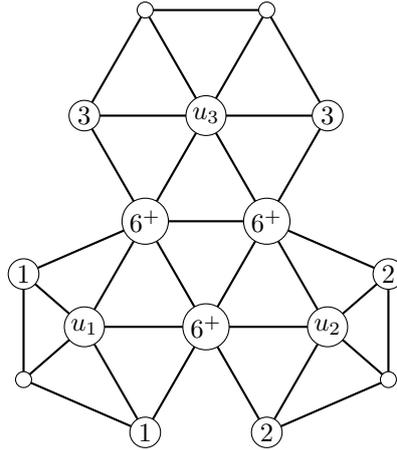


(d) A 7-vertex,  $v$ , with a 5-neighbor,  $u_1$ , and two 6-neighbors,  $u_2$  and  $u_3$ .



(e) A 7-vertex,  $v$ , with a 5-neighbor,  $u_2$ , and two 6-neighbors,  $u_1$  and  $u_3$ .

Figure 7: The five cases of Lemma 20.



(a) A 3-face  $v_1v_2v_3$ , such that the pairwise common neighbors of  $v_1, v_2, v_3$  have degrees 5, 5, and at most 6.

Figure 8: The key case of Lemma 21.

*Proof.* Lemma 6 yields  $12 \leq |N(\{u_1, u_2, u_3\})| \leq d(u_1) + d(u_2) + d(u_3) - 4 \leq 5 + 6 + 6 - 4 \leq 13$ . In Figure 7(a),  $|N(\{u_1, u_2, u_3\})| = 11$ . So, by symmetry, we assume that the vertices are arranged as in Figures 7(b,c) with all vertices distinct as drawn or as in Figures 7(d,e) with at most one pair of vertices identified.

First suppose the vertices are distinct as drawn. For Figures 7(b,c,d), we apply Lemma 4; for (b) and (c) we use  $k = 1$ , and for (d) we use  $k = 0$ . For Figure 7(e), we apply Lemma 16, using the vertices labeled 1 for  $M_1$  and the those labeled 2 for  $M_2$ . Now  $|N(\{u_1, u_2, u_3\})| \geq 14$  is a contradiction.

So, instead suppose that a single pair of vertices is identified in one of Figures 7(d,e). First consider (d). If a vertex labeled 1 is identified with another vertex, then we apply Lemma 13 using the vertices labeled 2 for the independent 2-set (vertices labeled 1 and 2 cannot be identified, since they are drawn at distance at most 3). Otherwise, the identified vertices must be those labeled 2 and 3 that are drawn at distance four. Now the vertices labeled 4 are pairwise at distance two, so must be an independent 4-set. Now we get a contradiction, by applying Lemma 11 using the vertices labeled 1 for the independent 2-set.

Finally, consider Figure 7(e), with a single pair of vertices identified. Again we apply Lemma 4, with  $k = 1$ . Since  $u_1$  has three possibilities for its pair of nonadjacent neighbors, and no neighbor of  $u_1$  appears in all three of these pairs,  $u_1$  satisfies condition (2). Similarly,  $u_3$  also satisfies condition (2).  $\square$

**Lemma 21.** *Let  $v_1, v_2, v_3$  be the corners of a 3-face, each a  $6^+$ -vertex. Let  $u_1, u_2, u_3$  be the other pairwise common neighbors of  $v_1, v_2, v_3$ , i.e.,  $u_1$  is adjacent to  $v_1$  and  $v_2$ ,  $u_2$  is adjacent to  $v_2$  and  $v_3$ , and  $u_3$  is adjacent to  $v_3$  and  $v_1$ . We cannot have  $|N(\{u_1, u_2, u_3\})| \leq 13$ . In particular, we cannot have  $d(u_1) = d(u_2) = 5$  and  $d(u_3) \leq 6$ .*

*Proof.* If the only pairwise common neighbors of the  $u_i$  are the  $v_i$ , then two  $u_i$  are 5-vertices and the third is a  $6^-$ -vertex. The case where the  $u_i$  have more pairwise common neighbors is nearly

identical, and we remark on it briefly at the end of the proof. So suppose that  $d(u_1) = d(u_2) = 5$  and  $d(u_3) = 6$ , as shown in Figure 8; the case where  $d(u_3) = 5$  is nearly identical. We will apply Lemma 4 with  $J = \{u_1, u_2, u_3\}$  and  $k = 0$ . Clearly,  $J$  is an independent set. Now we verify that each vertex of  $J$  satisfies condition (2). Since  $G$  has no separating 3-cycle, the two vertices in each pair with a common label (among  $\{1, 2, 3\}$ ) are distinct and nonadjacent. Similarly, the vertices with labels in  $\{1, 2, 3\}$  are distinct, since they are drawn at pairwise distance at most three, and  $G$  has no separating 3-cycle. Thus, we can apply Lemma 4, as desired.

In the more general case where the  $u_i$  have pairwise common neighbors in addition to the  $v_i$ , the argument above still shows that the vertices with labels in  $\{1, 2, 3\}$  are distinct. So again, we can apply Lemma 4 with  $k = 0$ .  $\square$

**Lemma 22.** *Let  $u_1$  be a 7-vertex with nonadjacent 5-vertices  $u_2$  and  $u_3$  each at distance two from  $u_1$ . A minimal  $G$  cannot have  $u_1$  and  $u_2$  with two common neighbors and also  $u_1$  and  $u_3$  with two common neighbors.*

*Proof.* This situation is shown in Figures 9(a,b,c), possibly with some vertices identified. Let  $J = \{u_1, u_2, u_3\}$ . Suppose that more than a single pair of vertices is identified, which implies  $|N(J)| \leq 11$ . If  $J$  is a non-maximal independent set, then this contradicts Lemma 6. So suppose that  $J$  is a maximal independent set. If  $|N(J)| \leq 10$ , then  $|G| \leq 13$ , so  $J$  is the desired independent set of size  $\frac{3}{13}|G|$ . Otherwise,  $|G| = 14$ , so exactly three vertices are identified. Now we find an independent 4-set. Either we can take the four vertices labeled 4, or the two labeled  $i$ , together with  $J \setminus \{u_i\}$ , for some  $i \in \{1, 2, 3\}$ . Thus, at most one pair of vertices drawn as distinct are identified.

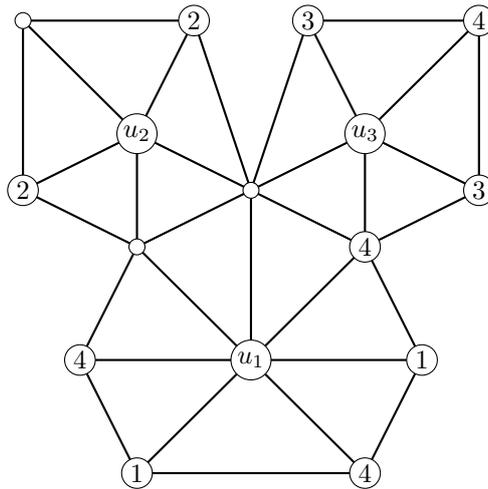
If all vertices labeled 2 or 3 are distinct as drawn, then we apply Lemma 16 and get a contradiction. By Lemma 6, the only other possibility is that exactly one pair of vertices is identified. Such a pair must consist of vertices labeled 2 and 3 that are drawn at distance four (otherwise we apply Lemma 4, with  $k = 1$ ). In Figure 9(a), this is impossible, since the two 5-vertices  $u_2$  and  $u_3$  would have two neighbors in common, violating Lemma 9.

Now we consider the cases shown in Figures 9(b,c) simultaneously. We apply Lemma 11 using the vertices labeled 1 for the independent 2-set. Let  $I_1$  be the set of vertices labeled 4. If  $I_1$  is independent, then we are done; so assume not. Recall that a vertex labeled 2 is identified with a vertex labeled 3.

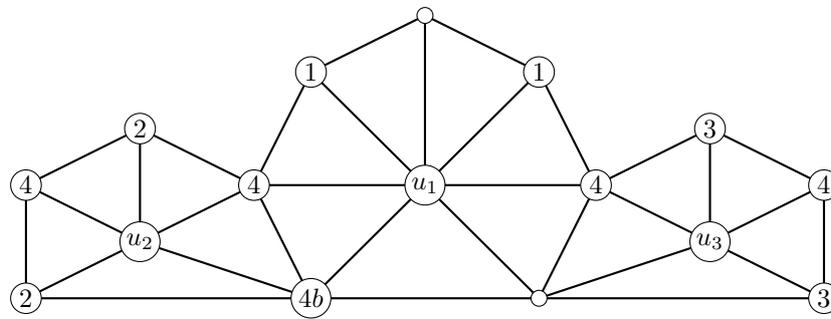
Suppose the vertices labeled 4 in  $N(u_2) \setminus N(u_1)$  and  $N(u_3) \setminus N(u_1)$  are not adjacent. Now by symmetry, we may assume that the vertex labeled 4 in  $N(u_1) \cap N(u_2)$  is adjacent to the vertex labeled 4 in  $N(u_3) \setminus N(u_1)$ . Let  $I_2$  be the set made from  $I_1$  by replacing the vertex labeled 4 in  $N(u_1) \cap N(u_2)$  with the vertex labeled  $4b$ . If  $I_2$  is independent, then we are done; so assume not. Now the vertex labeled  $4b$  must be adjacent to the vertex labeled 4 in  $N(u_3) \setminus N(u_1)$ , but this makes a separating 3-cycle (consisting of two vertices labeled 4 and one labeled  $4b$ ), a contradiction.

So, we may assume that the vertices labeled 4 in  $N(u_2) \setminus N(u_1)$  and  $N(u_3) \setminus N(u_1)$  are adjacent. Suppose the topmost vertex labeled 2 is identified with the topmost vertex labeled 3. Now again we are done; our independent 4-set consists of the two neighbors of  $u_1$  labeled 4, together with an independent 2-set from among the two leftmost and two rightmost vertices (by planarity, they cannot all four be pairwise adjacent).

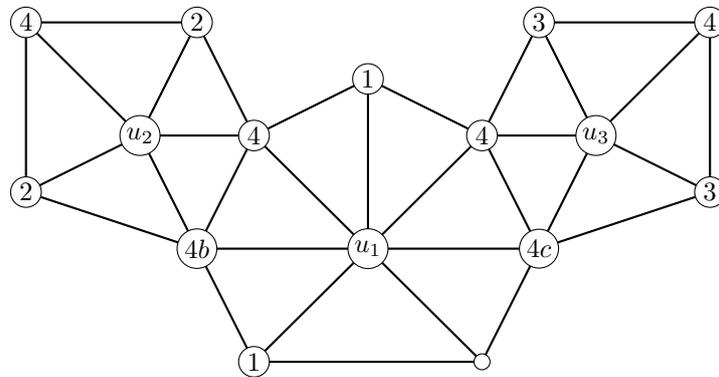
The only remaining possibility is that the bottommost vertex labeled 2 is identified with the bottommost vertex labeled 3 (since the two topmost vertices labeled 4 are adjacent). If



(a) Here  $u_2$  and  $u_3$  have a common neighbor in  $N(u_1)$ .



(b) Here  $u_2$  and  $u_3$  have adjacent neighbors in  $N(u_1)$ .



(c) Here  $u_2$  and  $u_3$  have neighbors at distance 2 in  $N(u_1)$ .

Figure 9: The cases of Lemma 22. The three possibilities for an independent 3-set  $\{u_1, u_2, u_3\}$  where  $d(u_1) = 7$ ,  $d(u_2) = d(u_3) = 5$ , and each of  $u_2$  and  $u_3$  has two neighbors in common with  $u_1$ .

we are in Figure 9(b), then the vertex labeled  $4b$  is a 5-vertex; since it shares two neighbors with  $u_3$ , another 5-vertex, we contradict Lemma 9. Hence, we must be in Figure 9(c). Now our independent 4-set consists of the two neighbors of  $u_1$  labeled  $4b$  and  $4c$ , together with an independent 2-set from among the four topmost vertices (again, by planarity, they cannot all be pairwise adjacent).  $\square$

**Lemma 23.** *Suppose that a minimal  $G$  contains a 7-vertex  $v$  with no 5-neighbor. Now  $v$  cannot have at least five 6-neighbors, each of which has a 5-neighbor.*

*Proof.* Suppose to the contrary. Denote the neighbors of  $v$  in clockwise order by  $u_1, \dots, u_7$ .

**Case 1:** Vertices  $u_1, u_2, u_3, u_4$  are 6-vertices, each with a 5-neighbor.

First, suppose that  $u_2$  and  $u_3$  have a common 5-neighbor,  $w_2$ . Consider the 5-neighbor  $w_1$  of  $u_1$ . By Lemma 9, it cannot be common with  $u_2$ ; similarly, the 5-neighbor  $w_4$  of  $u_4$  cannot be common with  $u_3$ . (We must have  $w_1$  and  $w_4$  distinct, since otherwise we apply Lemma 21 to  $\{u_1, u_4, w_2\}$ . Also, we must have  $w_1$  and  $w_4$  each distinct from  $w_2$ , since  $G$  has no separating 3-cycles.)

First, suppose that  $w_1$  has two common neighbors with  $u_2$ . If  $w_1 \not\leftrightarrow u_4$ , then we apply Lemma 19 to  $\{w_1, u_2, u_4\}$ ; so assume  $w_1 \leftrightarrow u_4$ . Now let  $J = \{u_1, u_4, w_2\}$ . Clearly,  $J$  is an independent 3-set. Also  $|N(J)| \leq 6 + 6 + 5 - 4 = 13$ , so we are done by Lemma 21. So  $w_1$  cannot have two common neighbors with  $u_2$ . Similarly,  $w_4$  cannot have two common neighbors with  $u_3$ . Hence,  $w_1 \leftrightarrow u_7$  and also  $w_4 \leftrightarrow u_5$ . Now we must have  $w_1 \leftrightarrow w_4$ ; otherwise we apply Lemma 22 to  $\{v, w_1, w_4\}$ . Similarly, we must have  $w_1 \leftrightarrow w_2$  and  $w_2 \leftrightarrow w_4$ ; these edges cut off  $w_4$  from  $u_1$ , so  $u_1 \not\leftrightarrow w_4$ . Since  $u_1$  and  $w_4$  are nonadjacent, but have a 5-neighbor in common, they must have two neighbors in common. So we apply Lemma 19 to  $\{u_1, u_3, w_4\}$ . Hence, we conclude that the common neighbor of  $u_2$  and  $u_3$  is not a 5-neighbor.

Since  $u_1$  and  $u_3$  are  $6^-$ -vertices, by Lemma 17, vertex  $u_2$  cannot have another  $6^-$ -vertex that is nonadjacent to  $u_1$  and  $u_3$ . Thus, a 5-neighbor of  $u_2$  must be a common neighbor of  $u_1$ ; call this 5-neighbor  $w_1$ . Similarly, the common neighbor  $w_4$  of  $u_3$  and  $u_4$  is a 5-vertex. We must have  $w_1 \leftrightarrow w_4$ , for otherwise we apply Lemma 22. We may assume that  $u_6$  is a 6-vertex. If not, then  $v$ 's five 6-neighbors, each with a 5-neighbor, are *successive*; so, by symmetry, we are in the case above, where  $u_2$  and  $u_3$  have a common 5-neighbor.

By planarity, either  $u_1 \not\leftrightarrow w_4$  or else  $u_4 \not\leftrightarrow w_1$ ; by symmetry, assume the former. Since  $u_1$  and  $w_4$  share a 5-neighbor (and are nonadjacent), they have two common neighbors. Now if  $u_6 \not\leftrightarrow w_4$ , then we apply Lemma 19 to  $\{u_1, u_6, w_4\}$ . Hence, assume  $u_6 \leftrightarrow w_4$ . This implies that  $u_4 \not\leftrightarrow w_1$ . Now, the same argument implies that  $u_6 \leftrightarrow w_1$ . Now let  $J = \{u_1, u_4, u_6\}$ . Lemma 6 gives  $12 \leq |N(J)| \leq 6 + 6 + 6 - 6 = 12$ . Thus the vertices of  $J$  have no additional pairwise common neighbors. Hence, we have an independent 2-set  $M_1$  in  $N(u_1) \setminus (N(u_4) \cup N(u_6))$ . Similarly, we have an independent 2-set  $M_4$  in  $N(u_4) \setminus (N(u_1) \cup N(u_6))$ . Now we apply Lemma 10 with  $J = \{u_1, u_4, u_6\}$  and  $S_1 = M_1 \cup \{u_1\}$  and  $S_2 = M_4 \cup \{u_4\}$ . In each case, we have  $\alpha(G[S_i \cup J]) \geq |M_i \cup \{u_{5-i}, u_6\}| = 4$ . This implies that  $|N(J)| \geq 13$ , a contradiction. Hence,  $v$  cannot have four successive 6-neighbors, each with a 5-neighbor.

**Case 2:** Vertices  $u_1, u_2, u_3, u_5, u_6$  are 6-vertices, each with a 5-neighbor.

Suppose that the common neighbor  $w_5$  of  $u_5$  and  $u_6$  is a 5-vertex. By symmetry (between  $u_1$  and  $u_3$ ) and Lemma 17, assume that the common neighbor  $w_2$  of  $u_2$  and  $u_3$  is a 5-vertex. If  $w_2 \not\leftrightarrow w_5$ , then we apply Lemma 22; so assume that  $w_2 \leftrightarrow w_5$ . If  $u_6 \not\leftrightarrow w_2$ , then apply Lemma 19 to  $\{u_6, u_1, w_2\}$  (note that  $u_6$  and  $w_2$  have two common neighbors, since they have a common 5-neighbor). So assume that  $u_6 \leftrightarrow w_2$ . Similarly, we assume that  $u_3 \leftrightarrow w_5$ , since otherwise we

apply Lemma 19 to  $\{u_3, u_1, w_5\}$ . Now consider the 5-neighbor  $w_1$  of  $u_1$ . By Lemma 9, it cannot be a common neighbor of  $u_2$  (because of  $w_2$ ). If it is a common neighbor of  $u_7$ , then we apply Lemma 22 to  $\{w_1, w_5, v\}$ ; note that  $w_1 \not\leftrightarrow w_5$ , since they are cut off by edge  $w_2u_6$ . Hence,  $w_1$  is neither a common neighbor of  $u_7$  nor of  $u_2$ . Now we apply Lemma 19 to  $\{u_2, w_1, w_5\}$ . Thus, we conclude that the common neighbor of  $u_5$  and  $u_6$  is not a 5-vertex.

Let  $x$  denote the common neighbor of  $u_5$  and  $u_6$ ; as shown in the previous paragraph,  $x$  must be a  $6^+$ -vertex. Suppose that the 5-neighbor  $w_5$  of  $u_5$  is also a neighbor of  $x$ . If  $w_5 \not\leftrightarrow u_1$ , then we apply Lemma 19 to  $\{u_6, u_1, w_5\}$ ; so assume that  $w_5 \leftrightarrow u_1$ . Now if the 5-neighbor  $w_6$  of  $u_6$  is also adjacent to  $x$ , then we apply Lemma 19 to  $\{u_5, w_6, u_3\}$  (we must have  $w_6 \not\leftrightarrow u_3$  due to edge  $w_5u_1$ ). So, by symmetry (between  $u_5$  and  $u_6$ ), we may assume that  $w_5 \leftrightarrow u_4$ . Now, by Lemma 17, the 5-neighbor  $w_2$  of  $u_2$  has is adjacent to either  $u_1$  or  $u_3$ . In either case, we must have  $w_2 \leftrightarrow w_5$ ; otherwise, we apply Lemma 22 to  $\{v, w_2, w_5\}$ . If  $w_6 \leftrightarrow u_7$ , then  $w_6 \leftrightarrow w_2$  and  $w_6 \leftrightarrow w_5$ ; otherwise, we apply Lemma 22 to  $\{v, w_6, w_2\}$  or  $\{v, w_6, w_5\}$ . Now we apply Lemma 19 to  $\{u_5, u_3, w_6\}$ . So instead  $w_6 \not\leftrightarrow u_7$ . Finally, we apply Lemma 19 to  $\{u_5, w_6, u_3\}$ . This completes the proof.  $\square$

## Acknowledgments

As we mentioned in the introduction, the ideas in this paper come largely from Albertson's proof [1] that planar graphs have independence ratio at least  $\frac{2}{9}$ . In fact, many of the reducible configurations that we use here are special cases of the reducible configurations in that proof. We very much like that paper, and so it was a pleasure to be able to extend Albertson's work. It seems that the part of his own proof that Albertson was least pleased with was verifying "unavoidability", i.e., showing that every planar graph contains a reducible configuration. In the introduction to Albertson [1], he wrote: "Finally Section 4 is devoted to a massive, ugly edge counting which demonstrates that every triangulation of the plane must contain some forbidden subgraph." This is one reason that we included in this paper a short proof of this same unavoidability statement, via discharging. We think Mike might have liked it.

The first author thanks his Lord and Savior, Jesus Christ.

## References

- [1] M. O. Albertson. A lower bound for the independence number of a planar graph. *J. Combinatorial Theory Ser. B*, 20(1):84–93, 1976.
- [2] K. Appel and W. Haken. Every planar map is four colorable. I. Discharging. *Illinois J. Math.*, 21(3):429–490, 1977.
- [3] K. Appel and W. Haken. Every planar map is four colorable. II. Reducibility. *Illinois J. Math.*, 21(3):491–567, 1977.
- [4] C. Berge. *Graphes et hypergraphes*. Dunod, Paris, 1970. Monographies Universitaires de Mathématiques, No. 37.
- [5] M. R. Garey and D. S. Johnson. The rectilinear Steiner tree problem is NP-complete. *SIAM J. Appl. Math.*, 32(4):826–834, 1977.