

Edge-coloring via fixable subgraphs

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Abstract

Many graph coloring proofs show that a minimal counterexample to the theorem being proved cannot contain certain configurations, and then that each graph under consideration contains some forbidden configuration. For that theorem, these configurations are *reducible*. We introduce a framework for proving reducibility for edge-coloring. *Fixability*, one form of reducibility, can be considered independent of a containing graph. This has two benefits: (i) we can formulate necessary conditions for fixability, and (ii) fixability is easily determined by a computer. The necessary condition of *superabundance* is sufficient for multistars and we conjecture that it is also sufficient for trees.

The computer can generate thousands of reducible configurations, but we have short proofs for few of these. Using these reducible configurations, we prove conjectures on edge-coloring for small maximum degree. This paper aims (i) to provide a common context for various known reducible configurations for edge-coloring and (ii) to spur development of methods for humans to understand what the computer already knows.

1 Introduction

Suppose we want to k -color a graph G . If we already have a k -coloring of an induced subgraph H of G , we might try to extend this coloring to all of G . We can view this task as coloring $G - H$ from lists (this is called *list-coloring*), where each vertex v in $G - H$ gets a list of colors formed from $\{1, \dots, k\}$ by removing all colors used on its neighborhood in H . Often we cannot complete just any k -coloring of H to all of G . Instead, we may need to modify the k -coloring of H to get a coloring we can extend. Given rules for how we may modify the k -coloring of H , we can view our original problem as the problem of list coloring $G - H$, where each vertex gets a list as before, but now we can modify these lists in certain ways.

As an example of this approach, the second author proved [Rabern 2014] a common generalization of Hall's marriage theorem and Vizing's theorem on edge-coloring. The present

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paper generalizes a special case of this result and puts it into a broader context. Since we often want to prove coloring results for all graphs having certain properties, and not just some fixed graph, we only have partial control over the outcome of a recoloring of H . For example, if we swap colors red and green in a component C of the red-green subgraph (that is, we perform a Kempe change), we may succeed in making some desired vertex red, but if C is somewhat arbitrary, we cannot precisely control what happens to the colors of its other vertices. We model this lack of control as a two-player game—we move by recoloring a vertex as we desire and then the other player gets a turn to muck things up. In the original context, where we want to color G , our opponent is the graph G ; more precisely, the embedding of $G - H$ in G is one way to describe a strategy for the second player. The general paradigm that we described above is for vertex coloring. In the rest of the paper, we consider only the special case that is edge-coloring (or, equivalently, vertex coloring line graphs).

All of our multigraphs are loopless. Let G be a multigraph, L a list assignment on $V(G)$, and $\text{pot}(L) = \bigcup_{v \in V(G)} L(v)$. An L -pot is a set X containing $\text{pot}(L)$. We typically let P denote an arbitrary L -pot. An L -edge-coloring is an edge-coloring π of G such that $\pi(xy) \in L(x) \cap L(y)$ for all $xy \in E(G)$; furthermore, we require $\pi(xy) \neq \pi(xz)$ for each vertex x and distinct neighbors y and z of x . For the maximum degree in G we write $\Delta(G)$, or simply Δ , when G is clear from context. For the edge-chromatic number of G we write $\chi'(G)$. We often denote the set $\{1, \dots, k\}$ by $[k]$.

2 Completing edge-colorings

Our goal is to convert a partial k -edge-coloring of a multigraph M into a k -edge-coloring of (all of) M . For a partial k -edge-coloring π of M , let M_π be the subgraph of M induced by the uncolored edges and let L_π be the list assignment on the vertices of M_π given by $L_\pi(v) = [k] - \{\tau \mid \pi(vx) = \tau \text{ for some edge } vx \in E(M)\}$.

Kempe chains give a powerful technique for converting a partial k -edge-coloring into a k -edge-coloring of the whole graph. The idea is to repeatedly exchange colors on two-colored paths until the uncolored subgraph M_π has an edge-coloring ζ from its lists, that is, such that $\zeta(xy) \in L_\zeta(x) \cap L_\zeta(y)$ for all $xy \in E(M_\pi)$. (One advantage of considering the special case that is edge-coloring is that every Kempe chain is either a path or an even cycle.) In this sense the original list assignment L_π on M_π is *fixable*. In the next section, we give an abstract definition of this notion that frees us from the embedding in the containing graph M . As we will see, computers enjoy this new freedom.

2.1 Fixable graphs

Thinking in terms of a two-player game is a good aid to intuition and we encourage the reader to continue doing so. However, a simple recursive definition is equivalent and has far less baggage. For distinct colors $a, b \in P$, let $S_{L,a,b}$ be all the vertices of G that have exactly one of a or b in their list; more precisely, $S_{L,a,b} = \{v \in V(G) \mid |\{a, b\} \cap L(v)| = 1\}$.

Definition 1. G is (L, P) -*fixable* if either

- (1) G has an L -edge-coloring; or

- (2) there are different colors $a, b \in P$ such that for every partition X_1, \dots, X_t of $S_{L,a,b}$ into sets of size at most two, there exists $J \subseteq [t]$ so that G is (L', P) -fixable, where L' is formed from L by swapping a and b in $L(v)$ for every $v \in \bigcup_{i \in J} X_i$.

The meaning of (1) is clear. Intuitively, (2) says the following. There is some pair of colors, a and b , such that regardless of how the vertices of $S_{L,a,b}$ are paired via Kempe chains for colors a and b (or not paired with any vertex of $S_{L,a,b}$), we can swap the colors on some subset J of the Kempe chains so that the resulting partial edge-coloring is fixable.

We write L -fixable as shorthand for $(L, \text{pot}(L))$ -fixable. When G is (L, P) -fixable, the choices of a, b , and J in each application of (2) determine a tree where all leaves have lists satisfying (1). The *height* of (L, P) is the minimum possible height of such a tree. We write $h_G(L, P)$ for this height and let $h_G(L, P) = \infty$ when G is not (L, P) -fixable.

Lemma 2.1. *If a multigraph M has a partial k -edge-coloring π such that M_π is $(L_\pi, [k])$ -fixable, then M is k -edge-colorable.*

Proof. Our proof is by induction on the height of $(L_\pi, [k])$. Choose a partial k -edge-coloring π of M such that M_π is $(L_\pi, [k])$ -fixable. If $h_{M_\pi}(L_\pi, [k]) = 0$, then (1) must hold for M_π and L_π ; that is, M_π has an edge-coloring ζ such that $\zeta(xy) \in L_\pi(x) \cap L_\pi(y)$ for all $xy \in E(M_\pi)$. Now $\pi \cup \zeta$ is the desired k -edge-coloring of M .

So we may assume that $h_{M_\pi}(L_\pi, [k]) > 0$. Choose colors $a, b \in [k]$ to satisfy (2) and give a tree of height $h_{M_\pi}(L_\pi, [k])$. Let H be the subgraph of M induced on all edges colored a or b , and let S be the vertices in M_π with degree exactly one in H . For each $x \in S$, let C_x be the component of H containing x . Since $|V(C_x) \cap S| \in \{1, 2\}$, the components of H give a partition X_1, \dots, X_t of S into sets of size at most two. Further, exchanging colors a and b on C_x has the effect of swapping a and b in $L_\pi(v)$ for each $v \in V(C_x) \cap S$. So we achieve the needed swapping of colors in the lists in (2) by exchanging colors on the components of H .

By (2) there is $J \subseteq [t]$ so that M_π is $(L', [k])$ -fixable, where L' is formed from L_π by swapping a and b in $L_\pi(v)$ for every $v \in \bigcup_{i \in J} X_i$. In fact, there is a J such that $(L', [k])$ has height less than that of $(L, [k])$. Let π' be the partial k -edge-coloring of M created from π by performing the color exchanges to create L' from L_π . By the induction hypothesis, M is k -edge-colorable. \square

2.2 Some examples

A graph G is Δ -edge-critical, or simply *edge-critical*, if $\chi'(G) > \Delta$, but $\chi'(G - e) \leq \Delta$ for every edge e . A *configuration* is a subgraph H , along with specified degrees $d_G(v)$ in the original graph for each vertex of H . A configuration H is *reducible* if there exists an edge $e \in E(H)$ such that whenever H appears as a subgraph (not necessarily induced) of a graph G , if $G - e$ has a Δ -edge-coloring, then so does G . A central tool for proving reducibility for edge-coloring is Vizing's Adjacency Lemma. For example, it yields a short proof of Vizing's Theorem that $\chi'(G) \leq \Delta + 1$ for every simple graph G .

Vizing's Adjacency Lemma (VAL). *Let G be a Δ -critical graph. If $xy \in E(G)$, then x is adjacent to at least $\max\{2, \Delta - d(y) + 1\}$ vertices of degree Δ .*

We can view VAL as giving conditions for the degrees of a vertex and its neighbors that yield a reducible configuration. Our goal now is to prove similar statements for larger configurations; we'd like a way to talk about configurations being reducible for k -edge-coloring. Lemma 2.1 gives us this with respect to a fixed partial k -edge-coloring π , but we want a condition independent of the particular coloring. Note that we have a lower bound on the sizes of the lists in L_π ; specifically, if π is a partial k -edge-coloring of a multigraph M , then $|L_\pi(v)| \geq k + d_{M_\pi}(v) - d_M(v)$ for every $v \in M_\pi$. This observation motivates the following definition.

Definition 2. If G is a graph and $f: V(G) \rightarrow \mathbb{N}$, then G is (f, k) -fixable if G is $(L, [k])$ -fixable for every L with $|L(v)| \geq k + d_G(v) - f(v)$ for all $v \in V(G)$.

This definition enables us to state our desired condition on reducible configurations for k -edge-coloring, which follows directly from Lemma 2.1.

Observation 1. If G is (f, k) -fixable, then G cannot be a subgraph of a $(k + 1)$ -edge-critical graph M where $d_M(v) \leq f(v)$ for all $v \in V(G)$.

Now we can talk about a graph G with vertices labeled by f being k -fixable. The computer is extremely good at finding k -fixable graphs. Combined with discharging arguments¹, this gives a powerful method for proving (modulo trusting the computer) edge-coloring results for small Δ . We'll see some examples of such proofs later; for now Figure 1 shows some 3-fixable graphs. A gallery of hundreds more fixable graphs is available at <https://dl.dropboxusercontent.com/u/8609833/Web/GraphData/Fixable/index.html>.

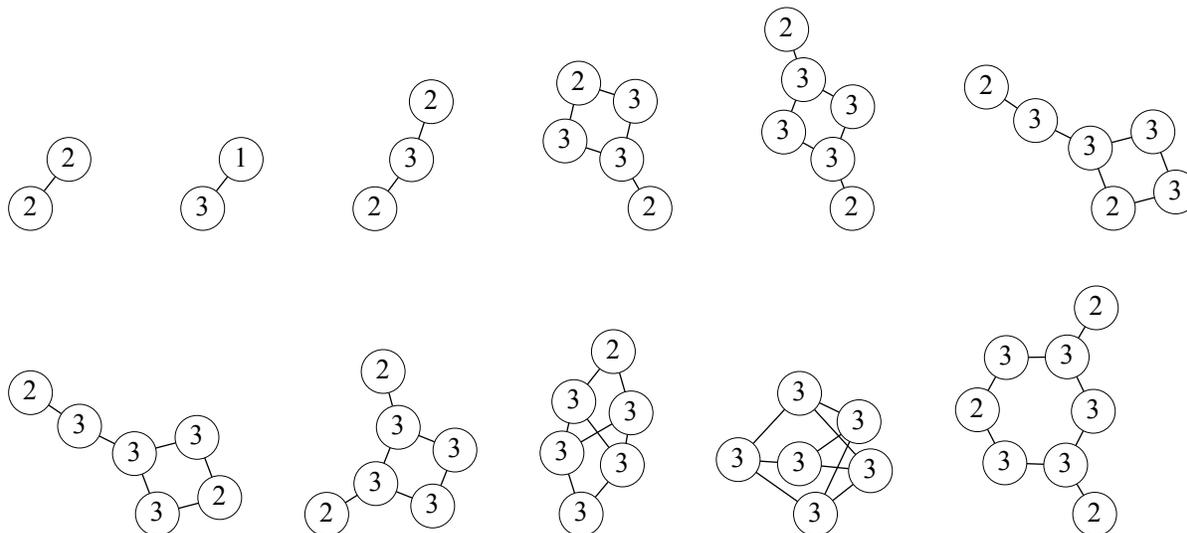


Figure 1: Some small 3-fixable graphs. (The label for each vertex specifies its degree in G .)

¹The discharging method is a counting technique commonly used in coloring proofs to show that the graph under consideration must contain a reducible configuration. For an introduction to this method, see [Cranston & West 2013].

The penultimate graph in Figure 1 is an example of the more general fact that a k -regular graph with $f(v) = k$ for all v is k -fixable precisely when it is k -edge-colorable. That the third graph in Figure 1 is reducible follows from Vizing's Adjacency Lemma.

2.3 A necessary condition

Since the edges incident to a vertex v must all get different colors, if G is (L, P) -fixable, then $|L(v)| \geq d_G(v)$ for all $v \in V(G)$. By considering the maximum size of matchings in each color, we get a more interesting necessary condition. For each $C \subseteq \text{pot}(L)$ and $H \subseteq G$, let $H_{L,C}$ be the subgraph of H induced by the vertices v with $L(v) \cap C \neq \emptyset$. When L is clear from context, we write H_C for $H_{L,C}$. If $C = \{\alpha\}$, we write H_α for H_C . For $H \subseteq G$, let

$$\psi_L(H) = \sum_{\alpha \in \text{pot}(L)} \left\lfloor \frac{|H_{L,\alpha}|}{2} \right\rfloor.$$

Each term in the sum gives an upper bound on the size of a matching in color α . So $\psi_L(H)$ is an upper bound on the number of edges in a partial L -edge-coloring of H . The pair (H, L) is *abundant* if $\psi_L(H) \geq \|H\|$ and (G, L) is *superabundant* if for every $H \subseteq G$, the pair (H, L) is abundant.

Lemma 2.2. *If G is (L, P) -fixable, then (G, L) is superabundant.*

Proof. Suppose instead that G is (L, P) -fixable and there is $H \subseteq G$ such that (H, L) is not abundant. We show that for all distinct $a, b \in P$ there is a partition X_1, \dots, X_t of $S_{a,b}$ into sets of size at most two, such that for all $J \subseteq [t]$, the pair (H, L') is not abundant, where L' is formed from L by swapping a and b in $L(v)$ for every $v \in \bigcup_{i \in J} X_i$. Since G can only be edge-colored from a superabundant list assignment, this contradicts that G is (L, P) -fixable.

Pick distinct colors $a, b \in P$. Let $S = S_{L,a,b} \cap V(H)$, let S_a be the $v \in S$ with $a \in L(v)$, and let $S_b = S \setminus S_a$. In the sum for $\psi_L(H)$, swapping a and b only effects the terms $\left\lfloor \frac{|S_a|}{2} \right\rfloor$ and $\left\lfloor \frac{|S_b|}{2} \right\rfloor$. So, if $\psi_L(H)$ is increased by the swapping, it must be that both $|S_a|$ and $|S_b|$ are odd, and after swapping they are both even. Say $S_a = \{a_1, \dots, a_p\}$ and $S_b = \{b_1, \dots, b_q\}$. By symmetry, we assume $p \leq q$. For each $i \in [p]$, let $X_i = \{a_i, b_i\}$. Since p and q are both odd, $q - p$ is even, so we get a partition by, for each $j \in \left[\frac{q-p}{2}\right]$, letting $X_{p+j} = \{b_{p+2j-1}, b_{p+2j}\}$. For any $i \in [p]$, swapping a and b in $L(v)$ for every $v \in X_i$ maintains $|S_a|$ and $|S_b|$. For any $j \in \left[\frac{q-p}{2}\right]$, swapping a and b in $L(v)$ for every $v \in X_{p+j}$ maintains the parity of $|S_a|$ and $|S_b|$. So no choice of J can increase $\psi_L(H)$. Thus, (H, L') is never abundant. \square

In particular, we conclude the following.

Corollary 2.3. *If G is (f, k) -fixable, then (G, L) is superabundant for every L with $L(v) \subseteq [k]$ and $|L(v)| \geq k + d_G(v) - f(v)$ for all $v \in V(G)$.*

Intuitively, superabundance requires the potential for a large enough matching in each color. If instead we require the existence of a large enough matching in each color, then we

get a stronger condition that has been studied before. For a multigraph H , let $\nu(H)$ be the number of edges in a maximum matching of H . For a list assignment L on H , let

$$\eta_L(H) = \sum_{\alpha \in \text{pot}(L)} \nu(H_\alpha).$$

Note that always $\psi_L(H) \geq \eta_L(H)$.

The following generalization of Hall's theorem was proved by Marcotte and Seymour [1990] and independently by Cropper, Gyárfás, and Lehel [2003]. By a *multitree* we mean a tree that possibly has edges of multiplicity greater than one.

Lemma 2.4 (Marcotte and Seymour). *Let T be a multitree and L a list assignment on $V(T)$. If $\eta_L(H) \geq \|H\|$ for all $H \subseteq T$, then T has an L -edge-coloring.*

The second author [Rabern 2014] proved that superabundance itself is also a sufficient condition for fixability, when we restrict our graphs to be multistars. This result immediately implies the *fan equation*, which is an extension of Vizing's Adjacency Lemma to multigraphs and a standard tool in proving reducibility for edge-coloring [see Stiebitz, Scheide, Toft, and Favrholt 2012, p. 19ff]. The proof for multistars uses Hall's theorem to reduce to a smaller star and one might hope that we could do the same for arbitrary trees, with Lemma 2.4 in place of Hall's theorem (thus giving a short proof that Tashkinov trees are elementary), but we haven't yet made this work.

2.4 Fixability of stars

When G is a star, superabundance implies fixability (provided that $|L(v)| \geq d_G(v)$ for each vertex v), and this result generalizes Vizing fans [Vizing 1976]. The second author [Rabern 2014] proved a common generalization of this and of Hall's theorem; below we reproduce the proof for the special case of edge-coloring. In the next section we define "Kierstead-Tashkinov-Vizing assignments" and show that they are always superabundant.

Theorem 2.5. *If G is a multistar, then G is L -fixable if and only if (G, L) is superabundant and $|L(v)| \geq d_G(v)$ for all $v \in V(G)$.*

Proof. Our strategy is simply to increase $\eta_L(G)$ if we can; if we cannot, then Hall's theorem allows us to reduce to a smaller graph. We can view this strategy as the following double induction. Suppose the theorem is false and choose a counterexample (G, L) minimizing $\|G\|$ and, subject to that, maximizing $\eta_L(G)$.

Let z be the center of the multistar G . Create a bipartite graph B with parts C and D , where C is the set of colors α that can be used on at least one edge, and D is the set of edges e with at least one color available on e , and a color α is adjacent to an edge e if α can be used on e . Note that $|C| = \eta_L(G)$.

First, suppose $|C| < \|G\|$. Since $|L(z)| \geq d_G(z) = \|G\|$, some color $\tau \in L(z)$ cannot be used on any edge. Suppose some color $\beta \in C$ can be used on at least three edges. Let zw be some edge that can use β . Since G is not L -fixable, there is $X \subseteq S_{L, \tau, \beta}$ with $w \in X$ and $|X| \leq 2$ such that G is not L' -fixable, where L' is formed from L by swapping τ and β in $L(v)$ for every $v \in X$. Since β can be used on at least three edges for L , it can be used

on at least one edge for L' . Further, τ can also be used on at least one edge for L' . Thus $\eta_{L'}(G) > \eta_L(G)$. Since (G, L') is still superabundant, this violates maximality of $\eta_L(G)$. Hence, each color $\beta \in C$ can be used on at most two edges. So, each color in C contributes at most one to $\psi_L(G)$.

Since $|C| < \|G\| \leq \psi_L(G)$, some color γ contributes at least 1 to $\psi_L(G)$, but is not in C . More precisely, some $\gamma \notin C$ satisfies $|G_\gamma - z| \geq 2$. Since G is not L -fixable, there is $X \subseteq S_{L,\tau,\gamma}$ with $z \in X$ and $|X| \leq 2$ such that G is not L' -fixable where L' is formed from L by swapping τ and γ in $L(v)$ for every $v \in X$. Since $\nu(G_{L,\tau}) = 0$ and $\nu(G_{L,\gamma}) = 0$ and $\nu(G_{L',\gamma}) = 1$, we have $\eta_{L'}(G) > \eta_L(G)$. Since (G, L') is still superabundant, this violates maximality of $\eta_L(G)$.

Hence, we must have $|C| \geq \|G\|$. In particular, $|N_B(C)| \leq |C|$ so we may choose a set of colors $C' \subseteq C$ such that C' is a minimal nonempty set satisfying $|N_B(C')| \leq |C'|$. If $|C'| \geq |N_B(C')| + 1$, then, for any $\rho \in C'$, we have $|C' - \rho| = |C'| - 1 \geq |N_B(C')| \geq |N_B(C' - \rho)|$, which contradicts the minimality of C' . Thus, $|C'| = |N_B(C')|$. Furthermore, by minimality of C' , every nonempty $C'' \subsetneq C'$ satisfies $|N_B(C'')| > |C''|$, so Hall's Theorem yields a perfect matching M between C' and $N_B(C')$.

For each color/edge pair $\{\alpha, zw\} \in M$, use color α on edge zw . Form G' from G by removing all the colored edges and then discarding any isolated vertices. Note that z lost exactly $|C'|$ colors from its list and also $d_{G'}(z) = d_G(z) - |C'|$, so $|L'(z)| = |L(z)| - |C'| \geq d_G(z) - |C'| = d_{G'}(z)$. Each other vertex $w \in V(G')$ satisfies $d_{G'}(w) = d_G(w)$ and $|L'(w)| = |L(w)|$, so $|L'(w)| \geq d_{G'}(w)$. Since G is not L -fixable and C' and $\text{pot}(L')$ are disjoint it must be that G' is not L' -fixable. For each $H \subseteq G'$, we have $\psi_{L'}(H) = \psi_L(H)$. For each color $\alpha \in C$, if $\alpha \in C'$, then $\lfloor |H_{L,\alpha}| / 2 \rfloor = 0$, since $E(H) \cap N_B(C') = \emptyset$. Similarly, if $\alpha \notin C'$, then each $v \in V(G')$ satisfies $\alpha \in L'(v)$ if and only if $\alpha \in L(v)$. Thus, H is abundant for L' precisely because H is abundant for L . But $\|G'\| < \|G\|$, so by minimality of $\|G\|$, G' is L' -fixable, a contradiction. \square

As shown by Rabern [2014], a direct consequence of Theorem 2.5 is the fan equation. This, in turn, implies most classical edge-coloring results including Vizing's Adjacency Lemma.

2.5 Kierstead-Tashkinov-Vizing assignments

Many edge-coloring results have been proved using a specific kind of superabundant pair (G, L) where superabundance can be proved via a special ordering. That is, the orderings given by the definition of Vizing fans, Kierstead paths, and Tashkinov trees (these structures are all standard tools in edge-coloring; definitions and more background are available in [Stiebitz, Scheide, Toft, and Favrholt 2012]). In this section, we show how superabundance follows easily from these orderings. For each vertex v , we write $E(v)$ for the set of edges incident to v .

A list assignment L on G is a *Kierstead-Tashkinov-Vizing assignment* (henceforth *KTV-assignment*) if for some edge $xy \in E(G)$, there is a total ordering ' $<$ ' of $V(G)$ such that

1. there is an edge-coloring π of $G - xy$ such that $\pi(uv) \in L(u) \cap L(v)$ for each edge $uv \in E(G - xy)$;
2. $x < z$ for all $z \in V(G - x)$;

3. $G[w \mid w \leq z]$ is connected for all $z \in V(G)$;
4. for each edge $wz \in E(G - xy)$, there is a vertex $u < \max\{w, z\}$ such that $\pi(wz) \in L(u) - \{\pi(e) \mid e \in E(u)\}$;
5. there are distinct vertices $s, t \in V(G)$ with $L(s) \cap L(t) - \{\pi(e) \mid e \in E(s) \cup E(t)\} \neq \emptyset$.

Lemma 2.6. *If L is a KTV-assignment on G , then (G, L) is superabundant.*

Proof. Let L be a KTV-assignment on G , and let $H \subseteq G$. We will show that (H, L) is abundant. Clearly it suffices to consider the case when H is an induced subgraph, so we assume this. Property (1) gives that $G - xy$ has an edge-coloring π , so $\psi_L(H) \geq \|H\| - 1$; also $\psi_L(H) \geq \|H\|$ if $\{x, y\} \not\subseteq V(H)$. Furthermore $\psi_L(H) \geq \|H\|$ if s and t from property (5) are both in $V(H)$, since then $\psi_L(H)$ gains 1 over the naive lower bound, due to the color in $L(s) \cap L(t)$. So $V(G) - V(H) \neq \emptyset$.

Now choose a vertex $z \in V(G) - V(H)$ that is smallest under $<$. Let $H' = G[w \mid w \leq z]$. By the minimality of z , we have $H' - z \subseteq H$. By property (2), $|H'| \geq 2$. By property (3), H' is connected and thus there is $w \in V(H' - z)$ adjacent to z . So, we have $w < z$ and $wz \in E(G) - E(H)$. Property (4) implies that there exists a vertex u with $u < \max\{w, z\} = z$ and $\pi(wz) \in L(u) - \{\pi(e) \mid e \in E(u)\}$. Since $u \in V(H' - z) \subseteq V(H)$, we again gain 1 over the naive lower bound on $\psi_L(H)$, due to the color in $L(u) \cap L(w)$. So $\psi_L(H) \geq \|H\|$. \square

2.6 The gap between fixability and reducibility

By abstracting away the containing graph, we may have lost some power in proving reducibility results. Surely we have when we only care about a certain class of graphs. For example, with planar graphs, not all Kempe path pairings are possible (if we add an edge for each pair, the resulting graph must be planar). But, possibly there are graphs that are reducible for all containing graphs but are not fixable. We could strengthen “fixable” in various ways, but we have not found the need to do so. One particular strengthening deserves mention, since it makes fixability more induction friendly.

Definition 3. G is (L, P) -subfixable if either

- (1) G is (L, P) -fixable; or
- (2) there is $xy \in E(G)$ and $\tau \in L(x) \cap L(y)$ such that $G - xy$ is L' -subfixable, where L' is formed from L by removing τ from $L(x)$ and $L(y)$.

Superabundance is a necessary condition for subfixability because coloring an edge cannot make a non-abundant subgraph abundant. The conjectures in the rest of this paper may be easier to prove with subfixable in place of fixable. That would really be just as good since it would give the exact same results for edge coloring.

3 Applications of small k -fixable graphs

In this section, we use k -fixable graphs to prove a few conjectures about 3-critical and 4-critical graphs. A k -vertex is a vertex of degree k , and a k -neighbor of a vertex v is a k -vertex adjacent to v .

3.1 The conjecture of Hilton and Zhao for $\Delta = 4$

For a graph G , let G_Δ be the subgraph of G induced by vertices of degree $\Delta(G)$. Vizing's Adjacency Lemma implies that $\delta(G_\Delta) \geq 2$ in a critical graph G . A natural question is whether or not this is best possible. For example, can we have $\Delta(G_\Delta) = 2$ in a critical graph G ? In fact, Hilton and Zhao have conjectured exactly when this can happen. Recall that a graph G is *class 1* if $\chi'(G) = \Delta$ and *class 2* otherwise. A graph G is *overfull* if $\|G\| > \lfloor \frac{|G|}{2} \rfloor \Delta(G)$. (The significance of overfull graphs is that they must be class 2, simply because they have more edges than can be colored by $\Delta(G)$ matchings, each of size $\lfloor \frac{|G|}{2} \rfloor$.) Let P^* denote the Peterson graph with one vertex deleted (see Figure 3).

Conjecture 3.1 (Hilton and Zhao). *A connected graph G with $\Delta(G_\Delta) \leq 2$ is class 2 if and only if G is P^* or G is overfull.*

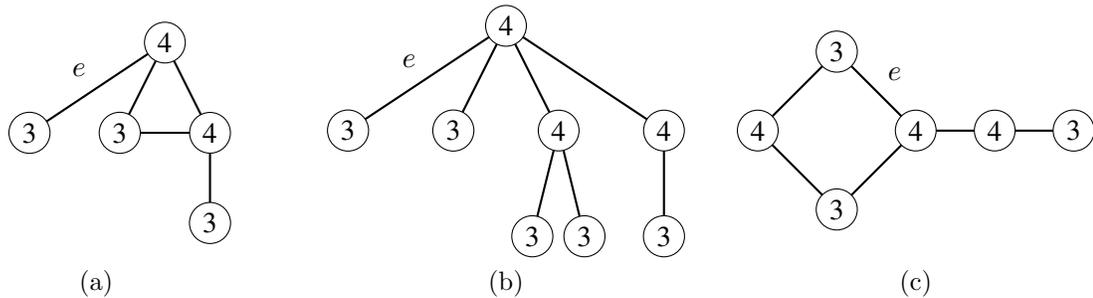


Figure 2: Each configuration is reducible by deleting edge e .

David and Gianfranco Cariolaro [Cariolaro & Cariolaro 2003] proved this conjecture when $\Delta = 3$. Here we prove it when $\Delta = 4$, but we omit the very long computer-generated proofs of the reducibility of the graphs in Figure 2. Since we do not include the reducibility proofs, we separate the proof into two parts. The first does not use the computer at all. Let \mathcal{H}_4 be the class of connected graphs with maximum degree 4, minimum degree 3, each vertex adjacent to at least two 4-vertices, and each 4-vertex adjacent to exactly two 4-vertices.

Lemma 3.2. *If G is a graph in \mathcal{H}_4 and G contains none of the three configurations in Figure 2 (not necessarily induced), then G is $K_5 - e$.*

Proof. Let G be a graph in \mathcal{H}_4 . Note that every 4-vertex in G has exactly two 3-neighbors and two 4-neighbors. Let u denote a 4-vertex and let v_1, \dots, v_4 denote its neighbors, where $d(v_1) = d(v_2) = 3$ and $d(v_3) = d(v_4) = 4$. When vertices x and y are adjacent, we write $x \leftrightarrow y$. We assume that G contains none of the configurations in Figure 2 and show that G must be $K_5 - e$.

First suppose that u has a 3-neighbor and a 4-neighbor that are adjacent. By symmetry, assume that $v_2 \leftrightarrow v_3$. Since Figure 2(a) is forbidden, we have $v_3 \leftrightarrow v_1$. Now consider v_4 . If v_4 has a 3-neighbor distinct from v_1 and v_2 , then we have a copy of Figure 2(c). Hence $v_4 \leftrightarrow v_1$ and $v_4 \leftrightarrow v_2$. If $v_3 \leftrightarrow v_4$, then G is $K_5 - e$. Suppose not, and let x be a 4-neighbor of v_4 .

Since G has no copy of Figure 2(c), x must be adjacent to v_1 and v_2 . This is a contradiction, since v_1 and v_2 are 3-vertices, but now each has at least four neighbors. Hence, we conclude that each of v_1 and v_2 is non-adjacent to each of v_3 and v_4 .

Now consider the 3-neighbors of v_3 and v_4 . If v_3 and v_4 have zero or one 3-neighbors in common, then we have a copy of Figure 2(b). Otherwise they have two 3-neighbors in common, so we have a copy of Figure 2(c). \square

Since $K_5 - e$ is overfull, the next theorem implies Hilton and Zhao's conjecture for $\Delta = 4$.

Theorem 3.3. *A connected graph G with $\Delta(G) = 4$ and $\Delta(G_\Delta) \leq 2$ is class 2 if and only if G is $K_5 - e$.*

Proof. Let G be as stated in the theorem. If G is class 2, then G has a 4-critical subgraph H . Since H is 4-critical, it is connected, and every vertex has at least two neighbors of degree 4, by VAL. Further, since $\Delta(H_\Delta) \leq \Delta(G_\Delta) \leq 2$, VAL implies that H has minimum degree 3. Thus, $H \in \mathcal{H}_4$. By Lemma 3.2, either H is $K_5 - e$ or H contains one of the configurations in Figure 2. By computer, each of these configurations is reducible and hence cannot be a subgraph of the 4-critical graph H . Thus H is $K_5 - e$. Let x_1, x_2 be the degree 3 vertices in H . Each x_i has three degree 4 neighbors in H and hence $d_G(x_i) \leq 3$ since $\Delta(G_\Delta) \leq 2$. That is, x_i has no neighbors outside H . Since G is connected, we must have $G = H = K_5 - e$. \square

3.2 Improved lower bounds on the average degree of 3-critical graphs and 4-critical graphs

Let P^* denote the Petersen graph with a vertex deleted (see Figure 3). Jakobsen [1973; 1974] noted that P^* is 3-critical and has average degree $2.\overline{6}$. He showed that every 3-critical graph has average degree at least $2.\overline{6}$, and asked whether equality holds only for P^* . We answered his question affirmatively [Cranston & Rabern 2015]. More precisely, we showed that every 3-critical graph other than P^* has average degree at least $2 + \frac{26}{37} = 2.\overline{702}$. The proof crucially depends on the fact that the three leftmost configurations in Figure 4 are reducible for 3-edge-coloring. As we noted previously [Cranston & Rabern 2015], by using the computer to prove reducibility of additional configurations, we can slightly strengthen this result. Specifically, every 3-critical graph has average degree at least $2 + \frac{22}{31} \approx 2.7097$ unless it is P^* or one other exceptional graph, the Hajós join of two copies of P^* . (For comparison, there exists an infinite family of 3-critical graphs with average degree less than 2.75.) This strengthening relies primarily on the fact that the rightmost configuration in Figure 4 is reducible, even if one or more pairs of its 2-vertices are identified. However, the simplest proof we have of this fact is computer-generated and fills about 100 pages.

Woodall [2008] conjectured that the average degree of every 4-critical graph is at least 3.6, which is best possible due to $K_5 - e$. We have proved this conjecture (modulo computer proofs of reducibility). However, the proof requires 39 reducible configurations, so we defer it to an appendix; even there, we omit the computer-aided reducibility proofs. Here, we give a brief outline to illustrate the technique.

Our proof uses the discharging method. Assume that G is a 4-critical graph. Each vertex begins with an initial charge, which is its degree. We redistribute the charge (without

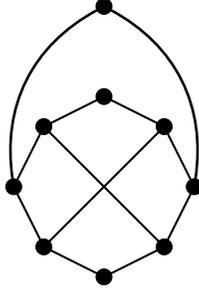


Figure 3: The Peterson graph with one vertex removed.

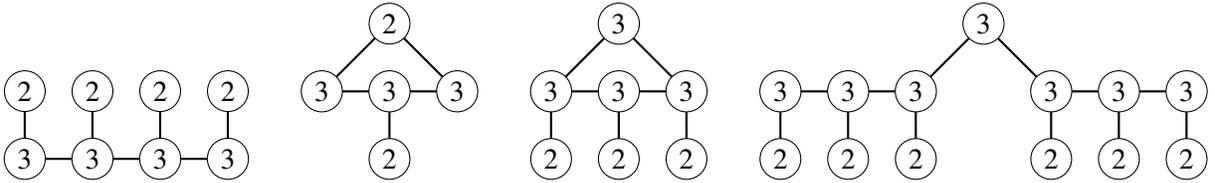


Figure 4: Four subgraphs forbidden from a 3-critical graph G .

changing its sum), with the goal that every vertex finishes with charge at least 3.6. If this is true, then G has the desired average degree. To redistribute charge, we successively apply the following 3 rules.

- (R1) Each 2-vertex takes .8 from each 4-neighbor.
- (R2) Each 3-vertex with three 4-neighbors takes .2 from each 4-neighbor. Each 3-vertex with two 4-neighbors takes .3 from each 4-neighbor.
- (R3) Each 4-vertex with charge in excess of 3.6 after (R2) splits this excess evenly among its 4-neighbors with charge less than 3.6.

By Vizing's Adjacency Lemma (VAL), each neighbor of a 2-vertex v is a 4-neighbor. Thus, v finishes with charge at least $2 + 2(.8) = 3.6$. Again by VAL, each 3-vertex v has at least two 4-neighbors. So v finishes with charge at least $3 + 3(.2)$ or $3 + 2(.3)$, both of which are at least 3.6.

It is also easy to check that each 4-vertex v finishes with charge at least 3.2; by VAL, v has at least two 4-neighbors, and if it has a 2-neighbor, then it has three 4-neighbors. So the remainder of the proof consists in showing that all 4-vertices that finish (R2) with charge less than 3.6 receive enough charge by (R3). The intuition is simple: if v has few low degree neighbors and neighbors of neighbors, then v gets enough charge; otherwise, v is contained in some reducible configuration, which contradicts our choice of G as 4-critical.

4 Superabundance sufficiency and adjacency lemmas

In the previous sections, we studied k -fixable graphs, which are reducible configurations for graphs with fixed maximum degree. Here we study a more general notion that behaves

similarly to Vizing Fans, Kierstead Paths, and Tashkinov Trees. Specifically, we consider graphs that are fixable for all superabundant list assignments.

4.1 Superabundant fixability in general

Definition 4. If G is a graph and $f: V(G) \rightarrow \mathbb{N}$ with $f(v) \geq d_G(v)$ for all $v \in V(G)$, then G is f -fixable if G is (L, P) -fixable for every L with $|L(v)| \geq f(v)$ for all $v \in V(G)$ and every L -pot P such that (G, L) is superabundant. If a graph G is f -fixable when $f(v) = d_G(v)$ for each v , then G is *degree-fixable*.

For example, Lemma 2.5 shows that multistars are degree-fixable. We have also found that the 4-cycle is degree-fixable.

Problem 1. *Classify the degree-fixable multigraphs (specifically, containment minimal ones).*

Since $f(v) \geq d_G(v)$, it is convenient to express the values of f as $d + k$ for a non-negative integer k ; this means $f(v) = d_G(v) + k$. When $k = 0$, we just write d , as in Figure 5. Looking at the trees in Figures 5, 6, and 7 we might conjecture that a tree is f -fixable whenever at most one internal vertex is labeled “ d ”. This conjecture continues to hold for many more examples.

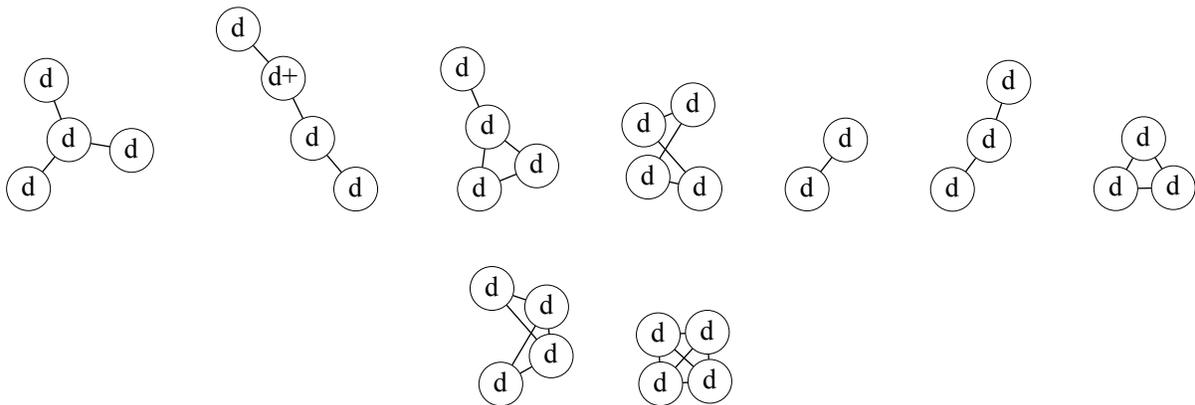


Figure 5: The fixable graphs on at most 4 vertices.

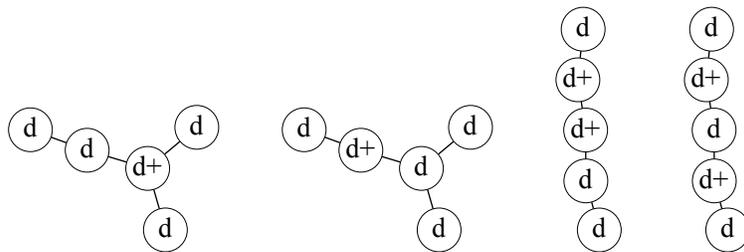


Figure 6: The fixable trees with maximum degree at most 3 on 5 vertices.

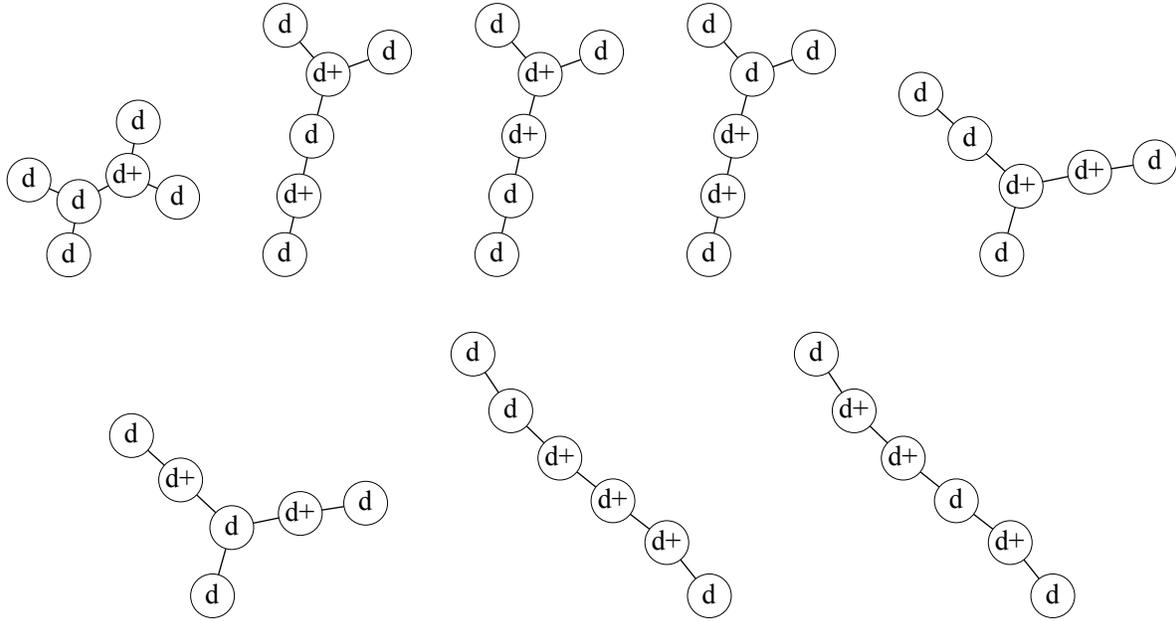


Figure 7: The fixable trees with maximum degree at most 3 on 6 vertices.

Conjecture 4.1. *A tree T is f -fixable if $f(v) = d_T(v)$ for at most one non-leaf v of T .*

Note that by Lemma 2.6, this would imply under the same degree constraints that Tashkinov trees are elementary (that is, each color is absent from at most one vertex of a Tashkinov tree). Can this be proved in the simpler case when the tree is a path? For paths of length 4, this was done by Kostochka and Stiebitz; in the next section we conjecture a generalization of their result to stars with one edge subdivided. One nice feature of the superabundance formulation is that since there is no need for an ordering as with Tashkinov trees, we can easily formulate results about graphs with cycles. The following is the most general thing we might think is true.

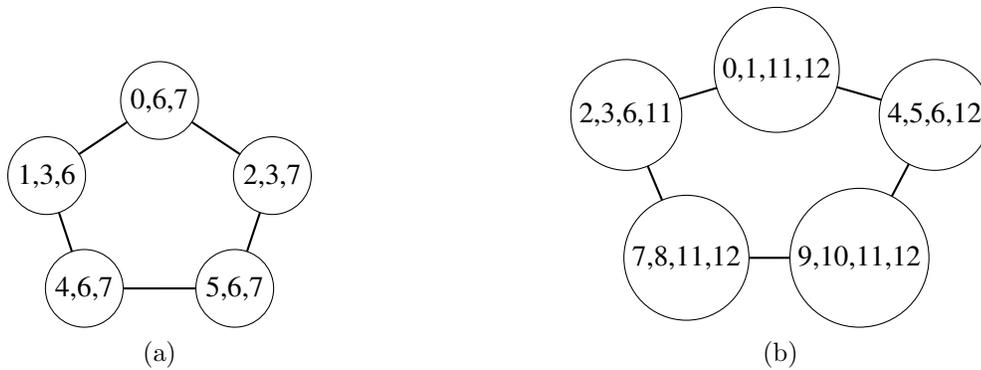


Figure 8: Counterexamples to Conjecture 4.2; the numbers at each vertex are its list of available colors.

Conjecture 4.2 (false). *A multigraph G is f -fixable if $f(v) > d_G(v)$ for all $v \in V(G)$.*

This conjecture is very strong and implies Goldberg’s conjecture [see Stiebitz, Scheide, Toft, and Favrholt 2012, p. 155ff], which is one of the major open problems in edge-coloring. Unfortunately, Conjecture 4.2 is false. We can make counterexamples on a 5-cycle as in Figure 8. We don’t yet have an intuitive explanation for why these are counterexamples, but in each case the computer has found a strategy preventing the 5-cycle from being colored.

One interesting consequence of these counterexamples is that C_5 is not f -fixable for *any* function f . (This contrasts with the case of (f, k) -fixable, since now increasing f need not increase $\psi(L)$.) Let L denote the lists in Figure 8(b). Given a function f , begin with L and add as many “singletons” (colors that appear in only one list) as needed to the lists so that each list is large enough; call these lists L' . Since L is superabundant, clearly so is L' . Now we play an “extra” game against the computer using L and we use the computer’s strategy in this extra game to inform our strategy for L' in the real game. If the colors chosen to swap in the real game are both in L , then we play with the computer’s strategy for the extra game. Any color $c \in L' \setminus L$ at a vertex v in the extra game is a singleton. Since only three colors of L are non-singletons, and $|L(v)| = 4$, each vertex in the extra game will always have a singleton in its list. Thus, we treat c like some other singleton c' at v in the extra game, and use the computer’s strategy from the extra game if c' had been chosen instead.

We have more questions than answers about Conjectures 4.1 and 4.2. For instance, what if in Conjecture 4.2 we only look at superabundant list assignments arising from an edge coloring of $G - e$, for some edge e ? The resulting conjecture is also stronger than Goldberg’s conjecture, and at present we have no counterexamples.

4.2 Stars with one edge subdivided

The following conjecture would generalize the “Short Kierstead Paths” of Kostochka and Stiebitz [see Stiebitz, Scheide, Toft, and Favrholt 2012, p. 46ff]. Parts (a) and (b) are special cases of Conjecture 4.1. We have a rough draft of a proof for part (a) and we suspect parts (b) and (c) will be similar, but our draft is long and detailed, and we are still hoping to find a clean proof, like that for stars. Recall that the fan equation implies the reducibility for k -edge-coloring of stars with certain specified degrees of the leaves. In this section we show that the truth of Conjecture 4.3 would imply a similar equation for stars with one edge subdivided.

Conjecture 4.3. *Let G be a star with one edge subdivided, where r is the center of the star, t the vertex at distance two from r , and s the intervening vertex. If L is superabundant and $|L(v)| \geq d_G(v)$ for all $v \in V(G)$, then G is L -fixable if at least one of the following holds:*

- (a) $|L(r)| > d_G(r)$; or
- (b) $|L(s)| > d_G(s)$; or
- (c) $\psi_L(G) > \|G\|$.

For a graph H and $v \in V(H)$, let $E_H(v)$ be the set of edges incident to v in H . Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$ and $G \subseteq Q$. For a $\Delta(Q)$ -edge-coloring π of

$Q - E(G)$, let $L_\pi(v) = [\Delta(Q)] - \pi(E_Q(v) - E(G))$ for all $v \in V(G)$. Graph G is a Ψ -subgraph of Q if there is a $\Delta(Q)$ -edge-coloring π of $Q - E(G)$ such that each $H \subsetneq G$ is abundant. Let $E_L(H) = |\{\alpha \in \text{pot}(L) \mid |H_{L,\alpha}| \text{ is even}\}|$ and $O_L(H) = |\{\alpha \in \text{pot}(L) \mid |H_{L,\alpha}| \text{ is odd}\}|$. Note that $\text{pot}(L) = E_L(G) + O_L(G)$.

Lemma 4.4. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$. If $G \subseteq Q$ and π is a $\Delta(Q)$ -edge-coloring of $Q - E(G)$ such that $\|G\| \geq \psi_L(G)$, then $|O_{L_\pi}(G)| \geq \sum_{v \in V(G)} \Delta(Q) - d_Q(v)$. Furthermore, if $\|G\| > \psi_L(G)$, then $|O_{L_\pi}(G)| > \sum_{v \in V(G)} \Delta(Q) - d_Q(v)$.*

Proof. The proof is a straightforward counting argument. For fixed degrees and list sizes, as $|O_L(G)|$ gets larger, $\psi_L(G)$ gets smaller (half as quickly). The details forthwith. Let $L = L_\pi$.

Since $\|G\| \geq \psi_L(G)$, we have

$$\|G\| \geq \sum_{\alpha \in \text{pot}(L)} \left\lfloor \frac{|G_{L,\alpha}|}{2} \right\rfloor = \sum_{\alpha \in \text{pot}(L)} \frac{|G_{L,\alpha}|}{2} - \sum_{\alpha \in O_L(H)} \frac{1}{2}. \quad (1)$$

Also,

$$\begin{aligned} \sum_{\alpha \in \text{pot}(L)} \frac{|G_{L,\alpha}|}{2} &= \sum_{v \in V(G)} \frac{\Delta(Q) - (d_Q(v) - d_G(v))}{2} \\ &= \sum_{v \in V(G)} \frac{d_G(v)}{2} + \sum_{v \in V(G)} \frac{\Delta(Q) - d_Q(v)}{2} \\ &= \|G\| + \sum_{v \in V(G)} \frac{\Delta(Q) - d_Q(v)}{2}. \end{aligned} \quad (2)$$

Now we solve for $\|G\| - \sum_{\alpha \in \text{pot}(L)} \frac{|G_{L,\alpha}|}{2}$ in (1) and (2), set the expressions equal, and then simplify. The result is (3).

$$|O_L(G)| \geq \sum_{v \in V(G)} \Delta(Q) - d_Q(v). \quad (3)$$

Finally, if the inequality in (1) is strict, then the inequality in (3) is also strict. \square

Again, let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$ and $G \subseteq Q$. If there is a $\Delta(Q)$ -edge-coloring π of $Q - E(G)$ such that each $H \subsetneq G$ is abundant, then G is a Ψ -subgraph of Q . The point of this definition is that if G is a Ψ -subgraph (and Conjecture 4.3(c) holds), then $\|G\| \geq \psi(G)$, so we can apply Lemma 4.4.

Conjecture 4.5. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$. Let H be a star with one edge subdivided; let r be the center of the star, t the vertex at distance two from r , and s the intervening vertex. If H is a Ψ -subgraph of Q , then there exists $X \subseteq N(r)$ with $V(H - r - t) \subseteq X$ such that*

$$\sum_{v \in X \cup \{t\}} (d_Q(v) + 1 - \Delta(Q)) \geq 0.$$

Moreover, if $\{r, s, t\}$ does not induce a triangle in Q , then

$$\sum_{v \in X \cup \{t\}} (d_Q(v) + 1 - \Delta(Q)) \geq 1.$$

Furthermore, if $d_Q(r) < \Delta(Q)$ or $d_Q(s) < \Delta(Q)$, then both lower bounds improve by 1.

Proof (assuming Conjecture 4.3). Let G be a maximal Ψ -subgraph of Q containing H such that G is a star with one edge subdivided. Let π be a coloring of $Q - E(G)$ showing that G is a Ψ -subgraph and let $L = L_\pi$.

We first show that $|E_L(G)| \geq d_Q(r) - d_G(r) - 1$ if rst induces a triangle; otherwise, $|E_L(G)| \geq d_Q(r) - d_G(r)$. Suppose rst does not induce a triangle; for an arbitrary $x \in N_Q(r) - V(G)$, let $\alpha = \pi(rx)$. Now consider adding x to G . By assumption, every $J \subsetneq G$ is abundant. Further, if $J \subsetneq G$ is abundant, then $J + x$ is also abundant. Thus, we only need to show that G is abundant. If $\alpha \in O_L(G)$, then adding x to G makes G abundant, since now r also has α in its list. This gives a larger Ψ -subgraph of the required form, which contradicts the maximality of G . Hence $\alpha \in E_L(G)$. Therefore, $|E_L(G)| \geq d_Q(r) - d_G(r)$ as desired. If rst induces a triangle, then we lose one off this bound from the edge rt .

By Conjecture 4.3(c), we have $\psi_L(G) \leq \|G\|$. Hence, by Lemma 4.4, we have $|O_L(G)| \geq \sum_{v \in V(G)} \Delta(Q) - d_Q(v)$. If rst does not induce a triangle, then

$$\begin{aligned} \Delta(Q) &\geq \text{pot}(L) \\ &= |E_L(G)| + |O_L(G)| \\ &\geq d_Q(r) - d_G(r) + \sum_{v \in V(G)} \Delta(Q) - d_Q(v) \\ &= \Delta(Q) - d_G(r) + \sum_{v \in V(G-r)} \Delta(Q) - d_Q(v) \\ &= \Delta(Q) + 1 + \sum_{v \in V(G-r)} \Delta(Q) - 1 - d_Q(v). \end{aligned} \tag{4}$$

Therefore, $\sum_{v \in V(G-r)} \Delta(Q) - 1 - d_Q(v) \leq -1$. Negating gives the desired inequality. If rst induces a triangle, then we lose one off the bound. Conjecture 4.3(a,b) gives the final statement. \square

5 Algorithm Overview

Here we describe the basic outline of our algorithm to test if a given graph G is k -fixable. To test if G is (L, P) -fixable for one L , we need to generate the two-player game tree. Doing this for every L would be a lot of work. With memoization, we can cut this down and get a reasonably efficient algorithm, but we can do much better by changing to a bottom-up strategy; that is, we do dynamic programming as follows.

1. Generate the set \mathcal{L} of all possible lists assignments L on G with $\text{pot}(L) \subseteq [k]$.

2. Create a set \mathcal{W} of *won* assignments, consisting of all $L \in \mathcal{L}$ such that G is L -colorable.
3. Put $\mathcal{L} := \mathcal{L} \setminus \mathcal{W}$.
4. For each $L \in \mathcal{L}$, check if there are different colors $a, b \in [k]$ such that for every partition X_1, \dots, X_t of $S_{L,a,b}$ into sets of size at most two, there exists $J \subseteq [t]$ so that $L' \in \mathcal{W}$, where L' is formed from L by swapping a and b in $L(v)$ for every $v \in \bigcup_{i \in J} X_i$. If so, add L to \mathcal{W} .
5. If step (4) modified \mathcal{W} , goto step (3).
6. G is k -fixable if and only if $\mathcal{L} = \emptyset$.

In step (1), we do not really want to generate *all* list assignments, just list assignments up to color permutation. To do this generation, we put an ordering on the set of list assignments and run an algorithm that outputs only the minimal representative of each color-permutation class. All the code lives in the GitHub repository of WebGraphs at <https://github.com/landon/WebGraphs>. Since a lot of this code is optimized for speed and not readability, a reference version is currently being built at <https://github.com/landon/Playground/tree/master/Fixability>.

6 Conclusion

Most work on proving sufficient conditions for k -edge-colorability relies on proving that various configurations are reducible. Although these reducibility proofs have common themes, they often feel ad hoc and are tailored to the specific theorem being proved. We have introduced the notion of fixability, which provides a unifying framework for many of these results. It also naturally leads to a number of conjectures which, if true, will likely increase greatly what we can prove about k -edge-coloring. The computer has provided significant experimental evidence for these conjectures (proving many specific cases), but offers little guidance toward proving them completely.

To conclude, we mention two consequences if Conjectures 4.3 and 4.1 are true. Vizing [1968] conjectured that every Δ -critical graph has average degree greater than $\Delta - 1$. For large Δ , the best lower bound is about $\frac{2}{3}\Delta$, due to Woodall [2007]. His proof relies on a new class of reducible configurations, which would be implied if both P_5 is fixable (a very special case of Conjecture 4.1) and Conjecture 4.3 is true. Another old conjecture of Vizing [1965] is that every n -vertex graph has independence number at most $\frac{1}{2}n$. The best upper bound known is $\frac{3}{5}n$, also due to Woodall [2011]. The proof is relatively short, but relies on the same reducible configurations just mentioned. Thus, proving Conjecture 4.5 and Conjecture 4.3 (even just for P_5) would put the best bounds for these two old problems into a much broader context.

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Appendix: Improved lower bound on the average degree of 4-critical graphs

Here we prove the $\Delta = 4$ case of Woodall's conjecture [Woodall 2008] on the average degree of a critical graph (modulo computer proofs of reducibility). We show all of the reducible configurations in Figures 9–47. A *board* is a list assignment for the vertices of the configuration. A board is *colorable* if the configuration can be colored immediately from that board. The computer shows that a configuration is reducible by considering all possible boards and verifying for each that some sequence of Kempe swaps leads to a colorable board. The *depth* of a board is the minimum number of Kempe swaps with which it can always reach a colorable board (each colorable board has depth 0). For each configuration we list the total number of boards, and also a vector, indexed from 0, where the i th coordinate is the number of boards of depth i .

Theorem 6.1. *If G is an edge-critical graph with maximum degree 4, then G has average degree at least 3.6. This is best possible, as shown by $K_5 - e$.*

Proof. We use discharging with initial charge $\text{ch}(v) = d(v)$ and the following rules.

- (R1) Each 2-vertex takes .8 from each 4-neighbor.
- (R2) Each 3-vertex with three 4-neighbors takes .2 from each 4-neighbor. Each 3-vertex with two 4-neighbors takes .3 from each 4-neighbor.
- (R3) Each 4-vertex with charge in excess of 3.6 after (R2) splits this excess evenly among its 4-neighbors with charge less than 3.6.

We write $\text{ch}^*(v)$ for the final charge of vertex v . We must show that every vertex v finishes with $\text{ch}^*(v) \geq 3.6$.

By VAL, each neighbor of a 2-vertex v is a 4-neighbor. Thus, $\text{ch}^*(v) = 2 + 2(.8) = 3.6$.

By VAL, each 3-vertex v has at least two 4-neighbors, so $\text{ch}^*(v) = 3 + 3(.2)$ or $\text{ch}^*(v) = 3 + 2(.3)$. In either case, $\text{ch}^*(v) = 3.6$.

Now we consider a 4-vertex v . Note that (R3) will never drop the charge of a 4-vertex below 3.6; thus, in showing that $\text{ch}^*(v) \geq 3.6$, we need not consider (R3).

Claim 1. *Let v be a 4-vertex with no 2-neighbor. If v is not on a triangle with degrees 3,3,4, then v finishes (R2) with charge at least 3.6; otherwise v finishes (R2) with charge 3.4.*

If v has at most one 3-neighbor, then v finishes (R2) with charge at least 3.7. By VAL, v has at most two 3-neighbors, so assume exactly two. If each receives charge .2 from v , then v finishes (R3) with charge 3.6, as desired. Otherwise, some 3-neighbor of v has its own 3-neighbor. Now G has a path with degrees 3,4,3,3, which is (C10), and hence reducible. Here we use that v does not lie on a triangle with degrees 3,3,4.

Claim 2. *Let v be a 4-vertex. If v has only 4-neighbors, then v splits its excess charge of .4 at most 2 ways in (R3) unless every 3^- -vertex within distance two of v is a 2-vertex; in that case v may split its excess at most 3 ways in (R3).*

From the previous claim, we see that a 4-vertex needs charge after (R2) only if it has a 2-neighbor or if it lies on a triangle with degrees 3,3,4. To prove the claim, we consider a 4-vertex v with 4-neighbors v_1, \dots, v_4 such that at least three v_i either have 2-neighbors or lie on triangles with degrees 3,3,4.

First suppose that at least two v_i lie on triangles with degrees 3,3,4. If two of these triangles are vertex disjoint, then we have (C13); otherwise, we have (C14). So we conclude that at most one v_i lies on a triangle with degrees 3,3,4. Assume that we have exactly one. Now we have either (C15) or (C16). Thus, we conclude that no v_i lies on a triangle with degree 3,3,4. If each v_i has a 2-neighbor, then we have (C17), (C18), or (C19). Thus, the claim is true.

Claim 3. *Every vertex other than a 4-vertex with a 2-neighbor finishes (R3) with at least 3.6.*

By Claim 1, we need only consider a 4-vertex v on a triangle with degrees 3,3,4; call its 3-neighbors u_1 and u_2 . Since v finishes (R2) with charge 3.4, we must show that in (R3) v receives charge at least .2. By Claim 2, this is true if v has any 4-neighbor with no 3-neighbors (note that its 4-neighbors cannot have 2-neighbors, since that yields (C20)). Thus, we assume that each 4-neighbor of v , call them u_3 and u_4 , has a 3-neighbor. If u_3 or u_4 has a 3-neighbor other than u_1 or u_2 , then the configuration is (C11), which is reducible. Thus, we may assume that u_3 is adjacent to u_1 and u_4 is adjacent to u_2 . Hence, each of u_3 and u_4 finishes (R2) with charge .1. It suffices to show that all of this charge goes to v in (R3). Thus, we need only show that no other neighbor of u_3 or u_4 needs charge after (R2). If it does, then we have the reducible configuration (C12), where possibly the rightmost 3 is a 2. Thus, v finishes with charge $4 - 2(.3) + 2(.1) = 3.6$, as desired.

Claim 4. *Every 4-vertex on a triangle with degrees 2,4,4 finishes (R3) with at least 3.6.*

Let v be a 4-vertex on a triangle with degrees 2,4,4, and let v_1 and v_2 be its 2-neighbor and 4-neighbor on the triangle. Let v_3 and v_4 be its other neighbors. By VAL, $d(v_3) = d(v_4) = 4$. We will show that each of v_3 and v_4 has only 4-neighbors and that each gives at least .2 to v in (R3). If v_3 or v_4 has a 3^- -neighbor, then we have (C21) or (C9), which are reducible; thus, each of v_3 and v_4 has only 4-neighbors. By Claim 2, vertex v_3 splits its charge of .4 at most two ways (thus, giving v at least .2) unless it gives charge to exactly three of its neighbors, each of which has a 2-neighbor. If this is the case, then we have (C22), (C23), or (C24), each of which is reducible. Thus, v_3 splits its charge at most two ways, and so gives v charge at least .2. By the same argument, v_4 gives v charge at least .2. Thus, v finishes (R3) with charge at least $4 - .8 + 2(.2) = 3.6$.

Now all that remains to consider is a 4-vertex v with a 2-neighbor and three 4-neighbors v_1, v_2, v_3 . Further, we may assume that v does not lie on a triangle with degrees 2,4,4. Also, we may assume that each v_i has no 2-neighbor; since v lies on no 2,4,4 triangle, this would yield a copy of (C9), which is reducible.

Claim 5. *If any v_i has two or more 3-neighbors, then v finishes with at least 3.6.*

Suppose that v_1 has two 3-neighbors (by VAL it can have no more). First, we note that neither v_2 nor v_3 has a 3-neighbor. If it's distinct from those of v_1 , then we have (C31);

otherwise, we have (C32). Thus, each of v_2 and v_3 finishes (R2) with excess charge .4. So, it suffices to show that each of v_2 and v_3 splits its excess charge at most two ways. By Claim 2, this is true unless v_2 (say) splits its charge among three 4-neighbors, each of which has a 2-neighbor. If v_2 does so, then we have (C33), (C34), or (C35), each of which is reducible. Thus, v finishes (R3) with charge at least $4 - .8 + 2(.2) = 3.6$.

Claim 6. *If any v_i has a 3-neighbor, which itself has a 3-neighbor, then v finishes with at least 3.6.*

Assume that v_1 has a 3-neighbor, which itself has a 3-neighbor. First, we note that neither v_2 nor v_3 has a 3-neighbor. If so, then we have one of (C42), (C43), or (C44). Thus, each of v_2 and v_3 finishes (R2) with excess charge .4. So, it suffices to show that each of v_2 and v_3 splits its excess charge at most two ways. By Claim 2, this is true unless v_2 (say) splits its charge among three 4-neighbors, each of which has a 2-neighbor. If v_2 does so, then we have (C45), (C46), or (C47), each of which is reducible. Thus, v finishes (R3) with charge at least $4 - .8 + 2(.2) = 3.6$.

Claim 7. *If no v_i has a 3-neighbor, then v finishes with at least 3.6.*

Since each v_i has only 4-neighbors, it finishes (R2) with excess charge .4. By Claim 2, it splits this charge at most 3 ways. Thus, v finishes with charge at least $4 - .8 + 3(.4/3) = 3.6$.

Claim 8. *If at least two vertices v_i have 3-neighbors, then v finishes with at least 3.6.*

Assume that v_1 and v_2 each have 3-neighbors. By Claim 5, each has exactly one 3-neighbor. By Claim 6, the 3-neighbors of v_1 and v_2 do not themselves have 3-neighbors. Thus, each of v_1 and v_2 finishes (R2) with charge exactly 3.8. Hence, in (R3), each splits its excess charge of .2 among its 4-neighbors that need charge. We show that in (R3) all of this charge goes to v . Suppose, to the contrary, that v_1 sends some of its charge elsewhere; this could be to (i) a 4-vertex with a 2-neighbor or (ii) a 4-vertex on a triangle with degrees 3,3,4. In (i), we have one of (C25)–(C28). In (ii), we have one of (C29) and (C30). All such configurations are reducible, which proves the claim.

Claim 9. *If exactly one vertex v_i has a 3-neighbor, then v finishes with at least 3.6.*

Assume that v_1 has exactly one 3-neighbor (and v_2 and v_3 have no 3-neighbors), so v_2 and v_3 each finish (R2) with excess charge .4. It suffices to show that either v_2 and v_3 each give v at least half of their charge in (R3) or else at least one of them gives v all its charge in (R3); assume not. By symmetry, we assume that v_2 splits its charge at least three ways in (R3) and v_3 splits its charge at least two ways. If v_3 gives charge to a 4-vertex with a 2-neighbor (in addition to v), then we have one of (C36)–(C40). So assume that v_3 gives charge to a 4-vertex on a triangle with degrees 3,3,4. By Claim 6, neither of these 3-vertices on the triangle is the 3-neighbor of v_1 . If we do not have (C37) or (C38), then we must have (C41), which is reducible. This proves the claim. \square



Figure 9: (C1) 18 total boards: In increasing depths (14, 1, 1, 1, 1).

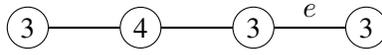


Figure 10: (C2) 26 total boards: In increasing depths (22, 1, 1, 1, 1).

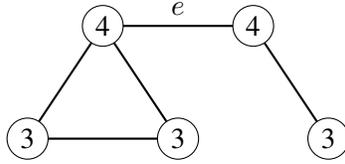


Figure 11: (C3) 84 total boards: In increasing depths (52, 20, 6, 4, 2).

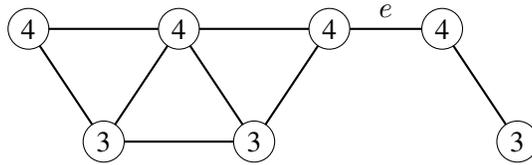


Figure 12: (C4) 32 total boards: In increasing depths (22, 5, 2, 2, 1).

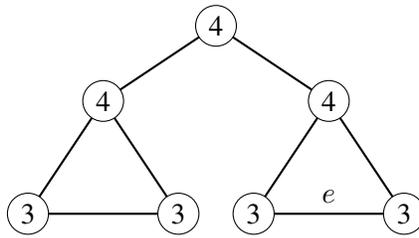


Figure 13: (C5) 652 total boards: In increasing depths (578, 36, 32, 6).

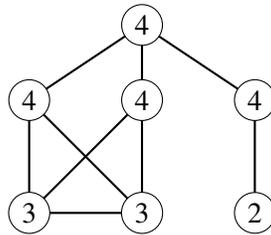


Figure 14: (C6) 72 total boards: In increasing depths (49, 20, 3).

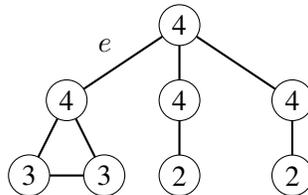


Figure 15: (C7) 3936 total boards: In increasing depths (2492, 750, 310, 266, 96, 14, 8).

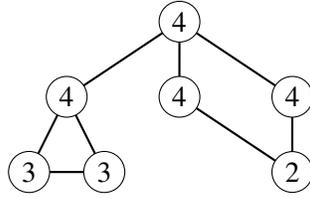


Figure 16: (C8) 391 total boards: In increasing depths (205, 163, 23).

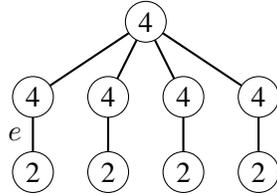


Figure 17: (C9) 6546 total boards: In increasing depths (5277, 279, 282, 285, 186, 135, 81, 21).

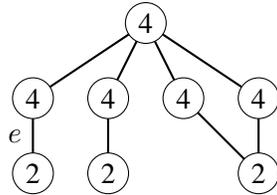


Figure 18: (C10) 526 total boards: In increasing depths (429, 51, 28, 16, 2).

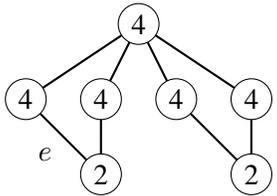


Figure 19: (C11) 41 total boards: In increasing depths (36, 5).

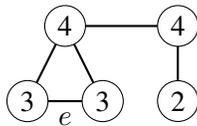


Figure 20: (C12) 55 total boards: In increasing depths (49, 4, 2).

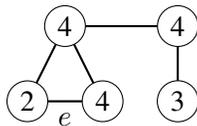


Figure 21: (C13) 22 total boards: In increasing depths (18, 1, 1, 1, 1).

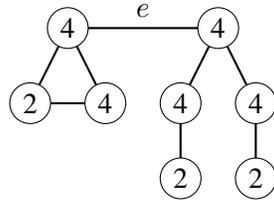


Figure 22: (C14) 1480 total boards: In increasing depths (868, 178, 140, 160, 92, 14, 18, 10).

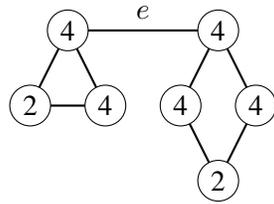


Figure 23: (C15) 114 total boards: In increasing depths (74, 28, 12).

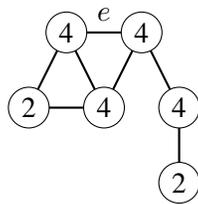


Figure 24: (C16) 62 total boards: In increasing depths (44, 15, 3).

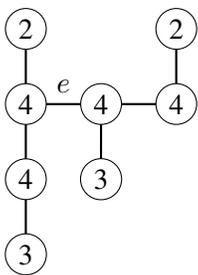


Figure 25: (C17) 7124 total boards In increasing depths (3648, 1081, 551, 579, 606, 448, 196, 15).

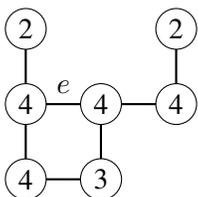


Figure 26: (C18) 1094 total boards In increasing depths (570, 248, 131, 96, 43, 6).

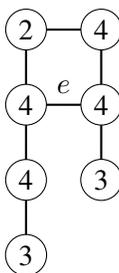


Figure 27: (C19) 534 total boards In increasing depths (297, 162, 19, 15, 17, 9, 2, 2, 2, 2, 2, 2, 3).

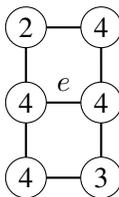


Figure 28: (C20) 85 total boards In increasing depths: (47, 27, 4, 4, 3).

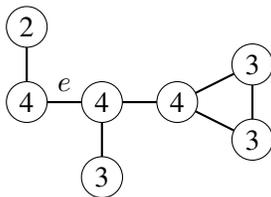


Figure 29: (C21) 1406 total boards: In increasing depths (790, 374, 188, 50, 4).

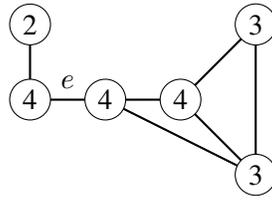


Figure 30: (C22) 72 total boards: In increasing depths (57, 14, 1).

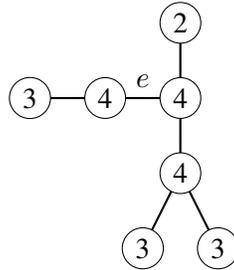


Figure 31: (C23) 2230 total boards In increasing depths (1158, 426, 267, 225, 100, 48, 6).

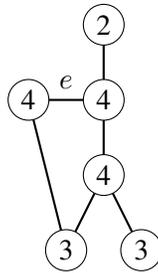


Figure 32: (C24) 335 total boards In increasing depths (182, 96, 44, 10, 2, 1).

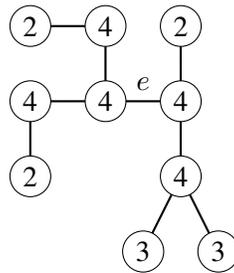


Figure 33: (C25) 108416 total boards: In increasing depths (58218, 17840, 12265, 10729, 5935, 2981, 422, 26).

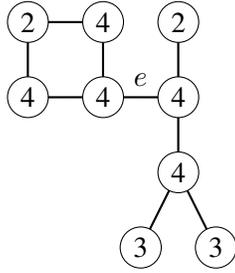


Figure 34: (C26) 8022 total boards: In increasing depths (4703, 1877, 1022, 420).

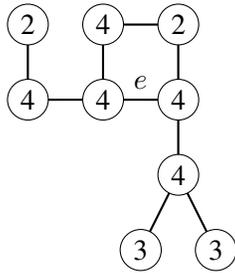


Figure 35: (C27) 8239 total boards: In increasing depths (4666, 2373, 922, 270, 8).

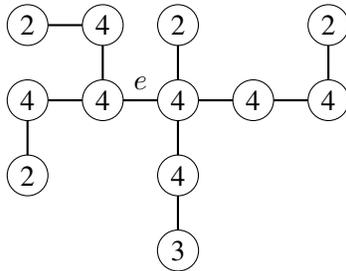


Figure 36: (C28) 579808 total boards: In increasing depths (292466, 64554, 28706, 24886, 24340, 23199, 23908, 26010, 25574, 23929, 15613, 5823, 792, 8).

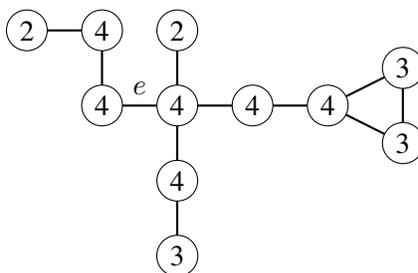


Figure 41: (C33) 134824 total boards: In increasing depths (66938, 19107, 11046, 9113, 8738, 8349, 7140, 3636, 737, 20).

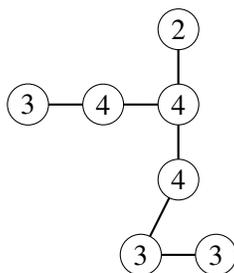


Figure 42: (C34) 3026 total boards: In increasing depth (1085, 380, 204, 147, 127, 143, 163, 145, 88, 90, 103, 28). (This configuration is unusual because there is no edge e such that it works to consider only near colorings for $G - e$; we need to go through a board that is a near coloring for a different missing edge.)

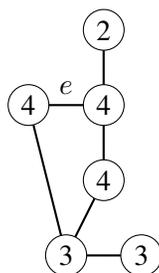


Figure 43: (C35) 142 total boards: In increasing depth (86, 49, 7).

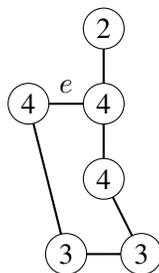


Figure 44: (C36) 328 total boards: In increasing depths (177, 50, 40, 41, 20).

