

Degeneracy and Colorings of Squares of Planar Graphs without 4-Cycles

Ilkyoo Choi* Daniel W. Cranston† Théo Pierron‡

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Abstract

We prove several results on coloring squares of planar graphs without 4-cycles. First, we show that if G is such a graph, then G^2 is $(\Delta(G) + 72)$ -degenerate. This implies an upper bound of $\Delta(G) + 73$ on the chromatic number of G^2 as well as on several variants of the chromatic number such as the list-chromatic number, paint number, Alon–Tarsi number, and correspondence chromatic number. We also show that if $\Delta(G)$ is sufficiently large, then the upper bounds on each of these parameters of G^2 can all be lowered to $\Delta(G) + 2$ (which is best possible). To complement these results, we show that 4-cycles are unique in having this property. Specifically, let S be a finite list of positive integers, with $4 \notin S$. For each constant C , we construct a planar graph $G_{S,C}$ with no cycle with length in S , but for which $\chi(G_{S,C}^2) > \Delta(G_{S,C}) + C$.

1 Introduction

The *square*, G^2 , of a graph G is formed from G by adding an edge vw for each pair of vertices, v and w , at distance two in G . It is easy to check that $\chi(G^2) \leq \Delta(G^2) + 1 \leq \Delta(G)^2 + 1$, and this bound can be tight, as when G is the 5-cycle or the Petersen graph (here χ and Δ denote, respectively, the chromatic number and maximum degree).¹ Even when $\Delta(G)$ is arbitrarily large, there exist constructions showing that this upper bound on $\chi(G)$ cannot be improved much. For example, when G is the incidence graph

*Department of Mathematics, College of Natural Sciences, Hankuk University of Foreign Studies (HUFS), Republic of Korea; ilkyoo@hufs.ac.kr; Supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2018R1D1A1B07043049), and also by the Hankuk University of Foreign Studies Research Fund.

†Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA, USA; dcranston@vcu.edu; This research is partially supported by NSA Grant H98230-15-1-0013.

‡Univ. Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR 5800, F-33400 Talence, France; tpierron@labri.fr

¹For simplicity, in this introduction we discuss only standard vertex coloring. But starting in Section 2 we consider degeneracy, and at the end of that section we mention multiple other graph coloring parameters.

of a projective plane,² we have $\chi(G^2) \approx \Delta(G)^2 - \Delta(G)$. However, for planar graphs, we have much better bounds on $\chi(G^2)$.

Recall that Euler's formula implies that every planar graph G is 5-degenerate. Coloring vertices greedily in the reverse of this degeneracy order [10],[6, Theorem 4.9] shows that $\chi(G^2) \leq 9\Delta(G)$. Refinements of this approach have led to successive improvements of this upper bound, culminating with the result of Molloy and Salavatipour [11] that $\chi(G^2) \leq \lceil \frac{5}{3}\Delta(G) \rceil + 78$. Havet et al. [9] also proved that $\chi(G^2) \leq \frac{3}{2}\Delta(G)(1+o(1))$, which strengthens the bound of [11] when $\Delta(G)$ is sufficiently large. Amini et al. [2] proved the same bound for all graphs embeddable in any fixed surface.

Every graph G satisfies $\chi(G^2) \geq \Delta(G) + 1$, and for planar graphs we might naively hope to prove a matching upper bound, or at least a bound of the form $\chi(G^2) \leq \Delta(G) + C$, for some constant C . However, for each $k \in \mathbb{Z}^+$, Wegner constructed a planar graph G_k with $\Delta(G_k) = k$ and $\chi(G_k^2) = \lfloor \frac{3}{2}k \rfloor + 1$; Figure 1 shows his construction. So to prove a bound of the form $\chi(G^2) \leq \Delta(G) + C$, we must restrict to some proper subset of planar graphs.

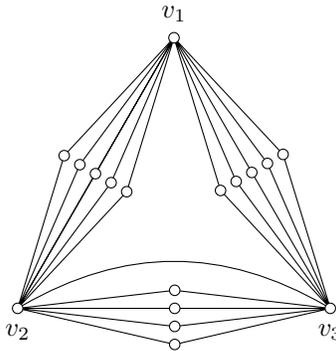


Figure 1: Wegner's construction

Wang and Lih [13] conjectured that, for each $g \geq 5$, there exists D_g such that if G is a planar graph with girth at least g and $\Delta(G) \geq D_g$, then $\chi(G^2) = \Delta(G) + 1$. This is true for $g \geq 7$ [4]. But it is false for girth 5 and 6 since, for each $k \geq 3$, there exists a planar graph G_k with $\Delta(G_k) = k$ and with girth 6 such that $\chi(G_k^2) = \Delta(G_k) + 2$ [4]. However, Dvořák et al. [8] proved a surprising complementary result: $\chi(G^2) \leq \Delta(G) + 2$ whenever G is a planar graph with girth 6 and $\Delta(G)$ sufficiently large. This work inspired analogous results for planar graphs with (i) girth 5 [3] and (ii) no 4-cycles or 5-cycles (though 3-cycles are allowed) [7]³. In each case the bound $\chi(G^2) \leq \Delta(G) + 2$ still holds (though the required lower bound on $\Delta(G)$ is larger).

The work above naturally leads to the following question. Exactly which cycle lengths can be forbidden from planar graphs to get a bound of the form $\chi(G^2) \leq \Delta(G) + C$? For a set \mathcal{S} of positive integers, let $\mathcal{G}_{\mathcal{S}}$ denote the family of planar graphs having no cycles with length in \mathcal{S} .

²This incidence graph G is $(k+1)$ -regular and bipartite with each part of size $k^2 + k + 1$. Since each pair of vertices within a part has a common neighbor, $\omega(G^2) = k^2 + k + 1$.

³Here we only hit the highlights. For a more detailed history of this problem, we recommend the introduction of [9] and [6, Conjecture 4.7 ff.].

Main Theorem. For a finite set \mathcal{S} there exists a constant $C_{\mathcal{S}}$ such that $\chi(G^2) \leq \Delta(G) + C_{\mathcal{S}}$ for all $G \in \mathcal{G}_{\mathcal{S}}$ if and only if $4 \in \mathcal{S}$.

We prove the Main Theorem in two parts. Immediately below we give a construction that proves the “only if” part. In Section 2 we handle the “if” part, the case when $4 \in \mathcal{S}$. In fact, we prove the stronger statement that the vertices of every graph $G \in \mathcal{G}_{\{4\}}$ can be ordered so that each vertex is preceded in the order by at most $\Delta(G) + 72$ of its neighbors in G^2 . Now the coloring result follows by coloring greedily. In Section 3, when $\Delta(G)$ is sufficiently large we strengthen our bound to $\chi(G^2) \leq \Delta(G) + 2$, which is sharp. This bound also holds for paint number, Alon–Tarsi number, and correspondence chromatic number (all defined at the end of Section 2).

Lemma 1.1. If $4, 2k \notin \mathcal{S}$, for some odd integer $k \geq 3$, then there does not exist a constant $C_{\mathcal{S}}$ such that $\chi(G^2) \leq \Delta(G) + C_{\mathcal{S}}$ for every $G \in \mathcal{G}_{\mathcal{S}}$.

Proof. Begin with a k -cycle and replace each edge vw with a copy of $K_{2,t}$, so that the two vertices of degree t replace v and w . The resulting graph, $G_{k,t}$ has maximum degree $2t$ and has cycles only of lengths 4 and $2k$. In every proper coloring of $G_{k,t}^2$, each color class contains at most $(k-1)/2$ vertices of degree 2 in $G_{k,t}$ (by the Pigeonhole Principle). Since $G_{k,t}$ has kt vertices of degree 2, we get $\chi(G_{k,t}^2) \geq kt / ((k-1)/2) = 2kt / (k-1) = 2t + 2t / (k-1) = \Delta(G) + 2t / (k-1)$. Given any constant C , we can choose t sufficiently large so that $2t / (k-1) > C$. \square

2 Graphs with no 4-cycles

Our goal in this section is to prove Theorem 2.1, below. First we need a few definitions. A k -vertex (resp. k^+ -vertex, k^- -vertex) is a vertex of degree equal to (resp. at least, at most) k ; a k -neighbor, of a vertex v , is an adjacent k -vertex. Analogously, we define k -face, k^+ -face, and k^- -face. We write $d(v)$ for the degree of a vertex v and $\ell(f)$ for the length of a face f . We write $N[v]$ to denote $N(v) \cup \{v\}$ and $N[S]$ for $\cup_{v \in S} N[v]$. We write $N^2(v)$ for the set of neighbors of v in G^2 . When the context could be unclear, we specify our meaning by using d_G , N_G , and N_G^2 . An order, σ , of $V(G)$ is *good for G* if each vertex, v , of G is preceded in σ by at most $\Delta(G) + 72$ vertices in $N^2(v)$. Following the approach of [5], we prove the degeneracy result below, which immediately implies the desired coloring bounds, by coloring greedily.

k -vertex
 k -neighbor
 k -face
 $N[v], N[S], N^2(v)$
good for G

Theorem 2.1. For every planar graph G with no 4-cycles, there exists a vertex order σ such that each vertex v is preceded in σ by at most $\Delta(G) + 72$ of its neighbors in G^2 .

Our proof of Theorem 2.1 is by discharging, with initial charge $d(v) - 4$ for each vertex v and $\ell(f) - 4$ for each face f . In the next section we discuss the discharging rules, but for now it is enough to note that we only need to give extra charge to 2-vertices, 3-vertices, and 3-faces. Here we prove that certain configurations are reducible; that is, they cannot appear in a minimal counterexample. In each case we assume that our minimal counterexample G contains such a configuration. We modify G to get a smaller graph G' (that is also planar and without 4-cycles), and which therefore has the desired vertex order, σ' . Finally, we modify σ' to get σ , a good vertex order for

G of $V(G)$. Each reducible configuration formalizes the intuition that every 2-vertex, 3-vertex, and 3-face of G must be near a vertex v of high degree. This is useful, since v has extra charge to share with nearby vertices and faces that need it.

Proof of Theorem 2.1. Suppose the theorem is false, and let G be a counterexample that minimizes the number of 3^+ -vertices and, subject to that, the number of edges. If G is disconnected, then we can get a good vertex order for each component (by minimality) and concatenate these to get a good order for G . Thus, G is connected. Similarly, if G has a 1-vertex v , then $G - v$ has a good order σ' and we can append v to σ' . So G has no 1-vertex. A vertex v is *big* if $d(v) \geq 10$, and v is *small* if $5 \leq d(v) \leq 9$. Note that $\Delta(G) \geq 10$, since otherwise each vertex has at most 9^2 neighbors in G^2 , so every vertex order shows that G is not a counterexample. big, small

2.1 Reducible Configurations

Key Lemma. *For an edge vw in G , if both v and w are not big, then at least one of v and w has at least two big neighbors.*

Proof. Suppose to the contrary that both v and w are not big, and that each has at most one big neighbor. By minimality, $G - vw$ has a good order, σ' . By deleting v and w from σ' , we get a good order (for G) of $V(G) - \{v, w\}$. Since v is not big and has at most one big neighbor, $|N^2(v)| \leq \Delta(G) + (10 - 1)(10 - 2)$. By symmetry, $|N^2(w)| \leq \Delta(G) + (10 - 1)(10 - 2)$. Thus, by appending v and w to the order, we get a good order for G , which is a contradiction. \square

Lemma 2.2. *If a 3-face f is incident with a 2-vertex, then the other two vertices on f must be big vertices.*

Proof. Let vw_1w_2 be a 3-face that is incident with a 2-vertex v . Suppose to the contrary that w_1 is not big. By minimality, $G - v$ has a good order, which is a good order for G of $V(G) - \{v\}$. Since w_1 is not big, $|N^2(v)| \leq \Delta(G) + 7$. Thus, we can append v to obtain a good order of G , which is a contradiction. \square

Lemma 2.3. *Every 3-face that is incident with two 3-vertices is also incident with a big vertex.*

Proof. Suppose that a 3-face is incident with two 3-vertices v_1, v_2 and a vertex w . Applying the Key Lemma to v_1v_2 shows that w must be big. \square

Lemma 2.4. *Every 3-vertex has a big neighbor.*

Proof. Let v be a 3-vertex with neighbors w_1, w_2, w_3 . Suppose to the contrary that every w_i is not big. Applying the Key Lemma to each edge vw_i shows that each w_i must be a 3^+ -vertex. Consider the graph G' formed from $G - v$ by adding a path of length two between each pair of neighbors of v . (Since each w_i is not big, we have $\Delta(G') = \Delta(G)$.) Since G' has fewer 3^+ -vertices, by minimality G' has a good order σ' , and σ' also is a good order for G of $V(G) - v$. Since each neighbor of v is small, $|N^2(v)| \leq 3 \cdot 9$. So appending v to σ' gives a good order for G , which is a contradiction. \square

Lemma 2.5. *If a 3-face f is incident with a 3-vertex v and at most one big vertex, then the neighbor of v that is not on f must be a big vertex.*

Proof. Let v be a 3-vertex on a 3-face vw_1w_2 and let x be the neighbor of v that is not on vw_1w_2 . Suppose to the contrary that both w_1 and x are not big. Applying the Key Lemma to edge vx shows that x is a 3^+ -vertex. Consider the graph G' formed from $G - v$ by adding paths of length two between x and w_1 and also between x and w_2 . So, G' has fewer 3^+ -vertices than G . By minimality, G' has a good order, σ' , which also is a good order for G of $V(G) - v$. Since v has at most one big neighbor, $|N^2(v)| \leq \Delta(G) + 16$. So appending v to σ' gives a good order for G , which is a contradiction. \square

2.2 Discharging

We use the initial charges $d(v) - 4$ for each vertex v and $\ell(f) - 4$ for each face f . Note that, by Euler's formula, the sum of these initial charges is -8 . Using the structural lemmas in Section 2.1, we redistribute this charge so that each vertex and face ends with nonnegative charge. However, this gives a contradiction, since a sum of nonnegative numbers cannot equal -8 . To redistribute charge, we use the following six discharging rules, applied in succession. (See Figure 2 for an illustration of the discharging rules.)

- (R1) Each edge takes $\frac{1}{5}$ from each incident 5^+ -face and $\frac{1}{10}$ from each incident big vertex⁴.
- (R2) If edge vw is incident to a 3-face f , then vw gives all its charge (received by (R1)) to f . Otherwise, vw distributes its charge equally among incident vertices x where $d(x) = \min\{d(v), d(w)\}$.
- (R3) Each big vertex gives $\frac{1}{2}$ to each neighbor.
- (R4) Each 3-vertex, 4-vertex, and small vertex gives $\frac{3}{5}$ to each 2-neighbor. If either v is a 4-vertex with at least two big neighbors or v is a small vertex, then v gives $\frac{1}{2}$ to each incident 3-face that is incident with a vertex other than v that is not big.
- (R5) Assume vertices v and w are big and the edge vw lies on a 3-face $vw x$. If x is a 4^- -vertex, then all charge given from v to w (and vice versa) by (R3) continues on to x . If x is a 5^+ -vertex, then all charge given from v to w (and vice versa) by (R3) continues on to face $vw x$.
- (R6) If a 3-vertex has an incident 3-face f with negative charge, then v gives its excess charge to f .

Now we show that each vertex and face ends with nonnegative charge, which yields the desired contradiction.

Each 5^+ -face f ends with charge $\ell(f) - 4 - \frac{1}{5}\ell(f) = \frac{4}{5}\ell(f) - 4 \geq 0$. Each edge receives charge by (R1) and gives it all away by (R2), so ends with 0. Consider a big vertex v . For each of its neighbors w , the charge that v gives to vw by (R1) is $\frac{1}{10}$ and to w by (R3) is $\frac{1}{2}$, for a total of $\frac{3}{5}$. So v ends with at least $d(v) - 4 - \frac{3}{5}d(v) = \frac{2}{5}d(v) - 4$; this is nonnegative, since $d(v) \geq 10$.

⁴A cut-edge takes $\frac{2}{5}$ from its incident face.

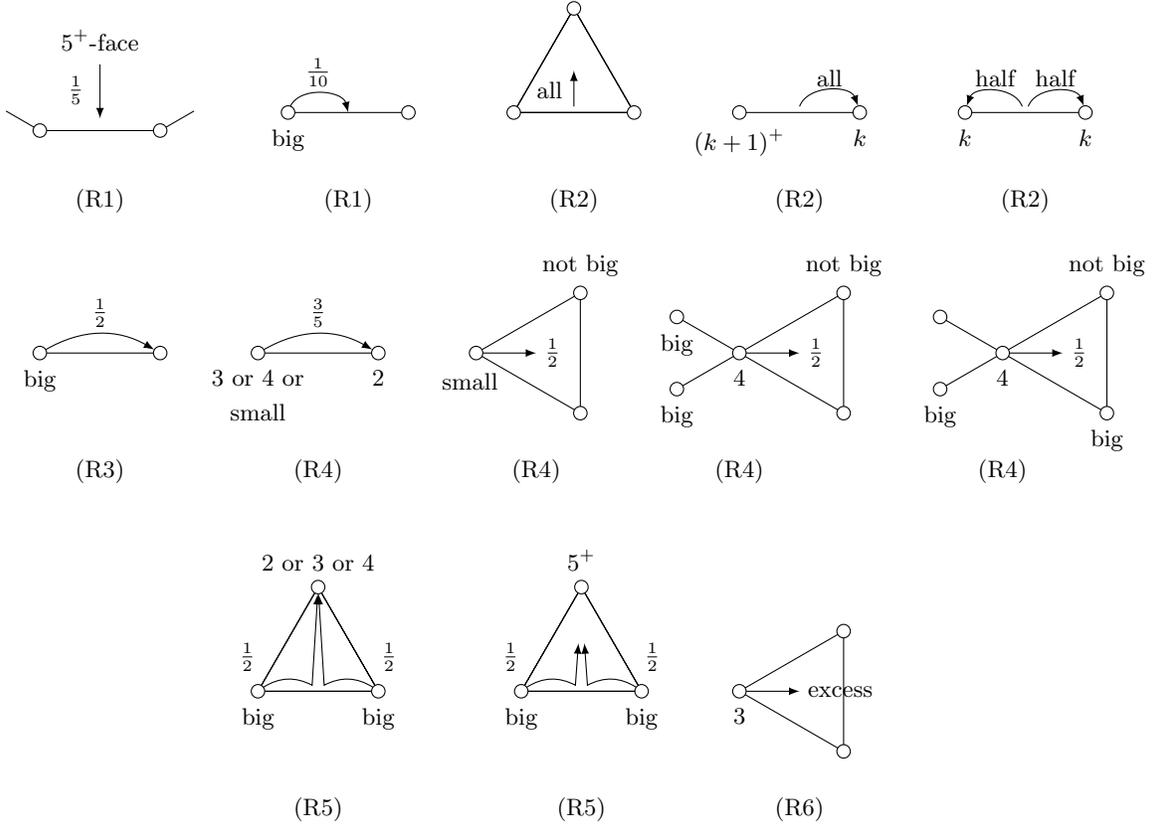


Figure 2: An illustration of the discharging rules.

Consider a small vertex v . Let $n_2(v)$ denote the number of 2-neighbors of v and $f_3(v)$ the number of 3-faces incident with v that are not incident with two big neighbors of v (that is, 3-faces that get $\frac{1}{2}$ from v). By (R4), v gives away $\frac{3}{5}n_2(v) + \frac{1}{2}f_3(v)$. Let $w_1, \dots, w_{d(v)}$ denote the neighbors of v . Suppose that v has a 2-neighbor. By the Key Lemma, v has at least two big neighbors, so $n_2(v) \leq d(v) - 2$. By Lemma 2.2, since v is small, if w_i is a 2-vertex, then neither face incident with vw_i receives charge from v . Thus, $n_2(v) + f_3(v) \leq d(v)$. By a more careful analysis, we will show that $n_2(v) + f_3(v) \leq d(v) - 2$. If $f_3(v) = 0$, then this inequality follows from $n_2(v) \leq d(v) - 2$ above. So assume $f_3(v) \geq 1$ and that v sends charge to some 3-face f'' with boundary that includes w_2vw_3 . Since G has no 4-cycles, neither the face preceding f'' around v nor the face following f'' is a 3-face, so neither of them receives charge from v ; call these faces f' and f''' . Further, some w_j other than w_2 and w_3 is big. Thus, we can pair the neighbors of v other than w_2 , w_3 , and w_j with the faces other than f' , f'' , and f''' such that each face is paired with an incident vertex and at most one element in each pair gets charge from v . This proves $n_2(v) + f_3(v) \leq d(v) - 2$. So, v ends with at least $d(v) - 4 + 2(\frac{1}{2}) - \frac{3}{5}(d(v) - 2) = \frac{2}{5}d(v) - \frac{9}{5}$; this is positive, since $d(v) \geq 5$. Now instead assume that v has no 2-neighbors. Since G has no 4-cycles, $f_3(v) \leq \lfloor \frac{d(v)}{2} \rfloor$.

$n_2(v), f_3(v)$

Thus, v ends with at least $d(v) - 4 - \frac{1}{2} \left\lfloor \frac{d(v)}{2} \right\rfloor$; this is nonnegative since $d(v) \geq 5$.

So, to complete the proof we only need to consider 3-faces, 2-vertices, 3-vertices, and 4-vertices.

Claim 2.6. *Every 2-vertex v that is on a 3-face vw_1w_2 ends with nonnegative charge.*

Proof. By Lemma 2.2, both w_1 and w_2 must be big. By (R3), v gets $\frac{1}{2}$ from each of w_1 and w_2 . And by (R5), v gets another $2(\frac{1}{2})$. So v ends with $2 - 4 + 4(\frac{1}{2}) = 0$. \diamond

Claim 2.7. *Every 2-vertex v that is not on a 3-face ends with nonnegative charge.*

Proof. Let w_1 and w_2 be the neighbors of a 2-vertex v . It suffices to show that v gets total charge at least 1 from w_1 and vw_1 , since by symmetry it also gets at least 1 from w_2 and vw_2 , so v ends with at least $2 - 4 + 2(1) = 0$. Applying the Key Lemma to vw_1 shows that w_1 either is big or is a 3^+ -vertex with two big neighbors. By (R1), vw_1 gets $\frac{2}{5}$ from incident faces and by (R2) vw_1 gives all this charge to v . So we only need to show that v gets at least $\frac{3}{5}$ from w_1 . If w_1 is a 3^+ -vertex that is not big, then w_1 gives $\frac{3}{5}$ to v by (R4). If w_1 is big, then it gives v charge $\frac{1}{2}$ by (R3), and gives edge vw_1 an extra $\frac{1}{10}$ by (R1), and all this charge goes to v by (R2). Thus, v gets $\frac{3}{5}$, as desired. \diamond

Claim 2.8. *Every 3-vertex v ends with nonnegative charge.*

Proof. By Lemma 2.4, v has a big neighbor w .

First suppose that v does not have a 2-neighbor. If vw is not on a 3-face, then by (R3) w gives v charge $\frac{1}{2}$, and by (R2) edge vw gives v charge $\frac{2}{5} + \frac{1}{10}$. So v ends with at least $3 - 4 + \frac{1}{2} + \frac{2}{5} + \frac{1}{10} = 0$. So assume v is on a 3-face and vz is on a 3-face for every big neighbor z of v . By Lemma 2.5, vertex v has at least two big neighbors, say w_1 and w_2 . Since each of vw_1 and vw_2 must be on a 3-face, and v has only a single incident 3-face, it must be vw_1w_2 . Now, by (R3) and (R5), v gets at least $4(\frac{1}{2})$. So v ends (R5) with at least $3 - 4 + 4(\frac{1}{2}) > 0$.

Now assume that v has a 2-neighbor x , which gets $\frac{3}{5}$ from v by (R4). Since $d(v) = 3$ and $d(x) = 2$, Lemma 2.2 implies that vx is not incident to any 3-face. Applying the Key Lemma to vx shows that v has two big neighbors, w_1 and w_2 . If vw_1w_2 is a 3-face, then each of w_1 and w_2 gives $\frac{1}{2} + \frac{1}{2}$ to v , by (R3) and (R5). So v ends with at least $3 - 4 - \frac{3}{5} + 4(\frac{1}{2}) > 0$. If vw_1w_2 is not a 3-face, then each of vw_1 and vw_2 gives $\frac{1}{2}$ to v , by (R1) and (R2). So v ends with at least $3 - 4 - \frac{3}{5} + 2(\frac{1}{2}) + 2(\frac{1}{2}) > 0$. \diamond

Claim 2.9. *Every 4-vertex v ends with nonnegative charge.*

Proof. Let $n_2(v)$ and $f_3(v)$ denote, respectively, the numbers of 2-neighbors and incident 3-faces that get charge from v by (R4).

Suppose v has no 2-neighbor. If v gives no charge to incident 3-faces by (R4), then v gives no charge at all, so v ends with at least $4 - 4 = 0$. If v does give charge to an incident 3-face by (R4), then (R4) implies that v has two big neighbors; by (R3), each big neighbor gives v charge $\frac{1}{2}$. Since G has no 4-cycles, v gives charge to at most two 3-faces. So v ends with at least $4 - 4 + 2(\frac{1}{2}) - 2(\frac{1}{2}) = 0$.

So assume v has a 2-neighbor, u . Applying the Key Lemma to uv shows that v has two big neighbors, w_1 and w_2 ; by (R3) each w_i gives v charge $\frac{1}{2}$. If vw_i is not on a

3-face, for some w_i , then by (R2) vw_i gives v charge $\frac{2}{5} + \frac{1}{10}$. Thus, v ends with at least $4 - 4 + 2(\frac{1}{2}) + (\frac{2}{5} + \frac{1}{10}) - 2 \cdot \frac{3}{5} > 0$. So we assume that each vw_i is on a 3-face. Since v has a 2-neighbor (which is not on a 3-face with v , by Lemma 2.2), and G has no 4-cycles, v has at most one incident 3-face. Since vw_1 and vw_2 are both on 3-faces, the 3-face must be vw_1w_2 . Because w_1 and w_2 are both big, v gives no charge to vw_1w_2 . So v ends with at least $4 - 4 + 4(\frac{1}{2}) - 2(\frac{3}{5}) > 0$. \diamond

Claim 2.10. *Every 3-face ends with nonnegative charge.*

Proof. Let $f = v_1v_2v_3$ be a 3-face, where $d(v_1) \leq d(v_2) \leq d(v_3)$. By (R1) each of v_1v_2, v_2v_3, v_3v_1 gets $\frac{1}{5}$ from its incident 5^+ -face, and by (R2) all of this charge goes to f . If f has two incident big vertices, then by (R1) edges v_1v_2, v_2v_3, v_3v_1 get in total an additional $\frac{4}{10}$. So f ends with at least $3 - 4 + \frac{3}{5} + \frac{4}{10} = 0$. If v_1 is a 2-vertex, then v_2 and v_3 are both big, by Lemma 2.2, and we are done, as above. So assume that v_1 is a 3^+ -vertex, and v_2 is not big. If some v_i is a small vertex or a 4-vertex with two big neighbors (which, by assumption, are not both incident to f), then v_i gives $\frac{1}{2}$ to f by (R4), so f ends with at least $3 - 4 + 3(\frac{1}{5}) + \frac{1}{2} > 0$. So we assume that f has at most one incident big vertex, and has no incident small vertex, and no incident 4-vertex with two big neighbors. Applying the Key Lemma to v_1v_2 shows that f must have an incident big vertex. Otherwise v_1 and v_2 are each 4^- -vertices with at most one big neighbor, a contradiction. Thus, we can assume that f has exactly one incident big vertex, and has no incident 2-vertex, small vertex, or 4-vertex with two big neighbors.

So assume that v_3 is big and that v_1 and v_2 are each either a 3-vertex or else a 4-vertex with no big neighbor other than v_3 . Applying the Key Lemma to v_1v_2 shows that v_1 must be a 3-vertex. Furthermore, at least one of v_1 and v_2 is a 3-vertex with a big neighbor w not on f ; by symmetry, assume this is v_1 . By (R3), w and v_3 each give v_1 charge $\frac{1}{2}$. Since edge wv_1 is not on a 3-face, by (R2) it gives v_1 charge $\frac{2}{5} + \frac{1}{10}$. So v_1 finishes (R5) with at least $3 - 4 + 2(\frac{1}{2}) + \frac{2}{5} + \frac{1}{10} = \frac{1}{5}$; by (R6) all of this charge continues on to f . So f ends with at least $3 - 4 + 3(\frac{1}{5}) + \frac{1}{2} > 0$. \diamond

This completes the proof of Theorem 2.1. \square

For completeness, we conclude this section with the definitions of Alon–Tarsi number, paint number and correspondence chromatic number, and the corollary that bounds these parameters for planar graphs with no 4-cycles. To denote the list-chromatic number of a graph G , we write $\chi_\ell(G)$.

An *eulerian digraph* is one in which each vertex has indegree equal to outdegree. For a digraph D , let $EE(D)$ and $EO(D)$ denote the numbers of eulerian subgraphs of D in which the number of edges is even and odd, respectively. A digraph D is *Alon–Tarsi* if $EE(D) \neq EO(D)$, and it is *k -Alon–Tarsi* if also each vertex has outdegree less than k . An *orientation* of a graph G is formed from G by directing each edge toward one of its endpoints. The *Alon–Tarsi number* of G , denoted $AT(G)$, is the smallest k such that some orientation of G is k -Alon–Tarsi. Note that every acyclic orientation D is Alon–Tarsi, since $EE(D) = 1 \neq 0 = EO(D)$; the only eulerian subgraph of D is the spanning edgeless graph. Suppose that G has degeneracy k , and σ is a vertex ordering witnessing this. By orienting each edge toward its endpoint that appears earlier in σ , we conclude that $AT(G) \leq k + 1$.

eulerian
digraph
Alon–Tarsi
 k -Alon–Tarsi
orientation
Alon–Tarsi
number

The paint number is defined using a two-player game. At round i , one player (Lister) chooses a set S_i of vertices and the other one (Painter) answers by coloring an independent subset of S_i with color i . The winning conditions depend on a fixed integer k : Lister wins if he presents a vertex on k rounds but Painter never colors it. Otherwise, Painter wins. The *paint number* $\chi_p(G)$ is the smallest integer k such that Painter has a winning strategy with parameter k . This problem can be seen as a generalization of list coloring, where the lists are not all known at the beginning of the coloring process (take S_i as the set of vertices whose lists contain color i). As shown by Schauz [12], each k -Alon–Tarsi graph is k -paintable. Thus, every k -degenerate graph G satisfies $\chi_p(G) \leq \text{AT}(G) \leq k + 1$.

paint number

Given a graph G and a function $f : V(G) \rightarrow \mathbb{N}$, an f -correspondence assignment C is given by a matching C_{vw} , for each $vw \in E(G)$, between $\{v\} \times \{1, \dots, f(v)\}$ and $\{w\} \times \{1, \dots, f(w)\}$. We say that each vertex x has $f(x)$ available colors. A k -correspondence assignment is an f -correspondence assignment where $f(v) = k$ for all $v \in V(G)$. Given an f -correspondence assignment C , a C -coloring is a function $\varphi : V(G) \rightarrow \mathbb{N}$ such that $\varphi(v) \leq f(v)$ for each $v \in V(G)$, and, for each edge $vw \in E(G)$, the pairs $(v, \varphi(v))$ and $(w, \varphi(w))$ are nonadjacent in C_{vw} . The *correspondence chromatic number* of G , denoted $\chi_{\text{corr}}(G)$, is the least integer k such that, for every k -correspondence assignment C of G , graph G admits a C -coloring. Note that if G is k -degenerate, then coloring greedily in some order witnessing this shows that $\chi_{\text{corr}}(G) \leq k + 1$. Thus, we have the following corollary of Theorem 2.1.

C -coloring

correspondence chromatic number

Corollary 2.11. *If G is a planar graph with no 4-cycles, then $\chi_{\text{corr}}(G^2) \leq \Delta(G) + 73$, $\chi_p(G^2) \leq \Delta(G) + 73$, and $\text{AT}(G^2) \leq \Delta(G) + 73$.*

3 Graphs with no 4-cycles and Δ large

In this section we show that the upper bounds in Corollary 2.11 can be strengthened to $\Delta(G) + 2$ when $\Delta(G)$ is sufficiently large. Initially, we just prove this upper bound for $\text{AT}(G^2)$, which also implies it for $\chi_p(G^2)$ and $\chi_\ell(G^2)$. In Section 3.6 we extend this result to $\chi_{\text{corr}}(G^2)$.

Theorem 3.1. *There exists Δ_0 such that if G is a plane graph with no 4-cycles and with $\Delta(G) \geq \Delta_0$, then G^2 is $(\Delta(G) + 2)$ -choosable. In fact, $\chi_p(G^2) \leq \text{AT}(G^2) \leq \Delta(G) + 2$.*

Let $\Delta_0 = 23769500^2 = 564989130250000$ and fix $k \geq \Delta_0$. We prove by contradiction that if G is a plane graph with no 4-cycles and with $\Delta(G) \leq k$, then G^2 is $(k + 2)$ -choosable. (By *plane graph*, we mean a planar graph with a fixed embedding in the plane. In particular, the neighborhood of each vertex is naturally endowed with a cyclic ordering.) For ease of exposition, we present the proof only for choosability, although it also works for paintability and Alon–Tarsi orientations. Most of the reducible configurations rely only on degeneracy, though at one point we use the kernel lemma.

plane graph

Assume the theorem is false and let G be a counterexample that minimizes $|E(G)| + |V(G)|$. Let L be an assignment of lists of size $k + 2$ to the vertices of G such that G^2 has no L -coloring. Throughout Section 3 we prove several structural lemmas, which

ultimately lead to a contradiction. We follow the same general approach as in [3], which considered planar graphs with girth at least 5; however, we need new ideas to handle the presence of triangles.

3.1 First Reducible Configurations

Lemma 3.2. *The graph G is connected and has minimum degree at least 2.*

Proof. Note that G is connected, since otherwise one of its components is a smaller counterexample. Now assume there exists a 1-vertex $v \in V(G)$. By the minimality of G , we can L -color $(G \setminus \{v\})^2$. Since $|L(v)| = k + 2$, and v has at most $1 + (k - 1)$ neighbors in G^2 , we can color v with a color not used on its neighbors in G^2 , which is a contradiction. \square

The next two lemmas essentially show that every vertex of G must be near a vertex of high degree. To formalize this, we use the following terminology: a vertex $v \in V(G)$ is *big* if $d(v) \geq \sqrt{k}$ and *small* otherwise. Denote by B and S the sets of big and small vertices. To refine the set S , we write S_i for the set of small vertices with exactly i big neighbors.

big, small

Remark 3.3. *In our figures in the rest of the paper, we draw small vertices as circles, and big vertices as squares. Further, we use black circles for vertices with all neighbors shown. So a white vertex could have more neighbors than those shown; in fact, it could also have edges (that are not drawn) to other vertices that are shown. For example, Figure 3 shows the configurations forbidden by Lemma 3.4.*

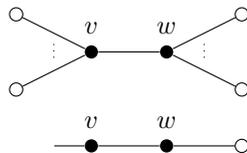


Figure 3: Forbidden configurations of Lemma 3.4.

Lemma 3.4. *For each edge $vw \in E(G)$, either $v \in N[B]$ or $w \in N[B]$. Further, if $d(v) = d(w) = 2$, then $v, w \in N[B]$.*

Proof. Assume to the contrary that some edge vw has $v, w \notin N[B]$. By minimality, we can L -color $(G - vw)^2$. We uncolor v and w . Since $v, w \notin N[B]$, both v and w have less than $\sqrt{k} \times \sqrt{k}$ colored neighbors in G^2 . Since $|L(v)| = |L(w)| = k + 2$, we can find distinct available colors for v and w .

Suppose instead that $d(v) = d(w) = 2$ and $v \in N[B]$ and $w \notin N[B]$. Again, by minimality we L -color $(G - vw)^2$, then uncolor v and w . Now v has at most $k + 1$ colored neighbors in G^2 , so v has an available color. As before, we can color w . This gives an L -coloring for G^2 , a contradiction. \square

Lemma 3.5. *If vw is an edge with $d(v) = d(w) = 2$, then v and w have no common neighbor.*

Proof. Assume there exists a triangle vwx with $d(v) = d(w) = 2$. By minimality, we can L -color $(G \setminus \{v, w\})^2$. Both v and w have $d(x) - 1 \leq k - 1$ colored neighbors in G^2 . So v and w each have at least 3 available colors, and thus we can color them both. \square

Lemma 3.6. *Let vx_1x_2 be a triangle of G such that some vertex $w \in S \setminus \{v, x_1, x_2\}$ has a common 2-neighbor with x_1 . If either (a) $d(x_2) = 2$ or (b) $d(x_2) = 3$ and w and x_2 have a common 2-neighbor, then $d(x_1) \geq 4$.*



Figure 4: Forbidden configurations of Lemma 3.6.

Proof. Let y_1 and y_2 denote the 2-neighbors of w common with x_1 and x_2 if they exist (in Case (a), only y_1 is defined). Assume that $d(x_1) = 3$. If $vw \in E(G)$, then wvx_1y_1 is a 4-cycle in G , a contradiction. So $vw \notin E(G)$. By assumption, $w \notin \{v, x_1, x_2, y_1\}$. So if $wx_2 \in E(G)$, then $wx_2x_1y_1$ is a 4-cycle in G , again a contradiction. Thus, $wx_2 \notin E(G)$. Since $d(x_1) = 3$ and $v, x_2, y_1 \in N(x_1)$, we must have $w \notin N(x_1)$. Since $N(y_1) = \{x_1, w\}$, also $vy_1 \notin E(G)$. So in both cases $wx_1, wx_2, wv, vy_1 \notin E(G)$. And in (b) also $vy_2 \notin E(G)$.

Let $S = \{x_1, x_2, y_1\}$ in Case (a), and $S = \{x_1, x_2, y_1, y_2\}$ in Case (b). By minimality, we L -color $(G \setminus S)^2$. For each $i \in \{1, 2\}$, the number of colored neighbors in G^2 of x_i is at most:

$$|\{v, w\}| + |N(v) \setminus \{x_1, x_2\}| \leq 2 + (k - 2) = k.$$

Thus, x_1 and x_2 both have at least 2 available colors, so we can color them. Further, for each $i \in \{1, 2\}$, the number of colored neighbors of y_i is at most

$$|\{v, w, x_1, x_2\}| + |N(w) \setminus \{y_i\}| \leq 4 + \sqrt{k} - 1 = \sqrt{k} + 3.$$

Therefore, y_1 and y_2 (if defined) both have $k - \sqrt{k} - 1$ available colors. Since k is large enough, we can color them to get an L -coloring for G^2 , a contradiction. \square

We combine Lemmas 3.4 and 3.6 to prove the reducibility of the bigger configuration shown in Figure 5.

Lemma 3.7. *Fix $v, w \in V(G)$ such that $w \in S$. Then the graph G cannot contain distinct vertices y_1, \dots, y_5 that are consecutive neighbors of w and that satisfy both conditions below; see Figure 5.*

1. Each y_i has degree two and has a common neighbor x_i with v .
2. For each $i \in \{1, \dots, 4\}$, each vertex inside cycle $vx_iy_iwy_{i+1}x_{i+1}$ is adjacent to v .

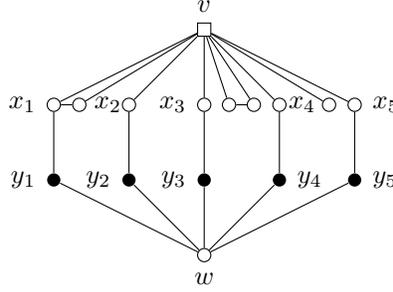


Figure 5: A possible configuration of Lemma 3.7.

Proof. We assume that G contains such a configuration and reach a contradiction, by showing that G contains a configuration forbidden by Lemma 3.6. Note that all x_i 's are distinct, since G contains no 4-cycle.

Below when we write a statement about x_i , we mean that it is true for each $i \in \{2, 3, 4\}$. Since $w \in S$, Lemma 3.4 implies that $d(x_i) \geq 3$. Because y_1, \dots, y_5 are consecutive neighbors of w , vertex x_i is not adjacent to w . Since G has no 4-cycle, x_i has at most one common neighbor with v . Thus $d(x_i) = 3$. Define z so that $N(x_3) = \{v, y_3, z\}$. If $z \in \{x_2, x_4\}$, then G contains the second configuration in Lemma 3.6, a contradiction. If z has a neighbor other than x_3 and v , then call it z' ; now z' is adjacent to v (by hypothesis 2), so vx_3zz' is a 4-cycle, a contradiction. Thus, z is a 2-vertex with $N(z) = \{x_3, v\}$. Now G contains the first configuration in Lemma 3.6, again a contradiction. \square

3.2 Outline of the proof

Recall that S is the set of small vertices, and S_i is the set of small vertices with exactly i big neighbors. Let G' denote the multigraph formed from G by suppressing every vertex of degree 2 in $S \setminus N[B]$, and then contracting every edge between S_1 and B . (Suppressing a 2-vertex v means deleting v and adding an edge between its two neighbors.) Note that G' may contain loops. For example, there is a loop in G' around a vertex u if u is a big vertex in G and there is a triangle uvw with $v, w \in S_1$. We say that a vertex of G *disappears* when constructing G' if it is either a suppressed vertex, or a vertex in S_1 .

Let G'' denote the multigraph formed from G' by removing every loop, and let G''' denote the underlying multigraph of G'' , i.e., the multigraph formed from G'' by deleting the minimal number of edges to remove all faces of length 2. Note that G''' can have parallel edges. For example, suppose v and w have parallel edges, say e_1 and e_2 , in G' . If some vertices are embedded inside and outside of the cycle e_1e_2 , then in G''' vertices v and w still have parallel edges, with those same vertices embedded inside and outside of the cycle e_1e_2 . However, G''' cannot have faces of length 2.

An r -region of G'' is a set $\{f_1, \dots, f_r\}$ of r pairwise distinct faces of length 2 such that:

- For $1 \leq i < r$, f_i shares one edge with f_{i+1} . (We say that the f_i 's are *consecutive*.)

- All the f_i 's have the same vertices b_1, b_2 on their boundary, where b_1 and b_2 are distinct vertices of B .

Note that each of the faces in an r -region is constructed from some cycle of G when we apply the construction rules above. By extension, an r -region of G is the subgraph of G induced by the vertices of these cycles, together with those lying on the inside of those cycles. (We often simply write *region*, when the specific value of r is less important.) When R is an r -region of G , we say that r is the size of R , and we denote by $V(R)$ the set of vertices appearing on all faces of R , excluding b_1 and b_2 .

r -region of G
region
 $V(R)$

To reach a contradiction, we prove the following two propositions.

Proposition 3.8. G contains an r -region with $r \geq \frac{\sqrt{k}}{50} - 37$.

Proposition 3.9. G does not contain any r -region for $r \geq 475353$.

Our contradiction now comes quickly. These propositions give that $\frac{\sqrt{k}}{50} - 37 < 475353$. This inequality implies $k < 23769500^2$, contradicting the hypothesis $k \geq \Delta_0 = 23769500^2$.

We will devote a subsection to the proof of each proposition: Subsection 3.4 for Proposition 3.8 and Subsection 3.5 for Proposition 3.9. In Subsection 3.3, we prove structural lemmas about the regions in G .

3.3 Structure of Regions

We now classify each edge of G' based on its corresponding path (or cycle) in G . An edge e in G' corresponds to a path or cycle $x_1 \cdots x_n$ in G if $e = x_1 x_n$ and for each $i \in \{2, \dots, n-1\}$, one of the following holds:

corresponds to
a path or cycle
 $x_1 \cdots x_n$ in G

- x_i is a 2-vertex in G and $x_{i-1}, x_{i+1} \in S$, or
- $x_i \in S_1$ and either x_{i-1} or x_{i+1} lies in B .

Due to the construction of G' , for every loop (resp. non-loop edge) e of G' , there is a unique cycle (resp. path) $x_1 \cdots x_n$ in G corresponding to e (with possibly $n = 2$). Note that we used here that the suppressed 2-vertices are not in $N[B]$, hence every contracted edge (between S_1 and B) is between two adjacent vertices in G .

The following lemma ensures that every edge (resp. loop) of G' corresponds to a short path (resp. cycle) of G . It also gives a classification of all the possible such paths (resp. cycles), depicted in Figure 6.

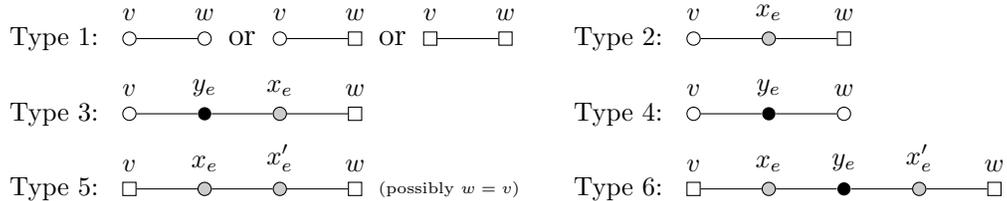


Figure 6: The six types of paths/cycles in G that create edges in G' (gray vertices lie in S_1).

Lemma 3.10. *Each edge $e = vw$ of G' corresponds to a path or a cycle in G for which exactly one of the following six conditions holds (up to exchanging v with w). If e satisfies condition i below (for some $i \in [6]$), then we say that e has type i . If $v \in S$, then e has one of types 1–4. If e is a loop of G' , then e has type 5. Finally if $v, w \in B$, then e has type 1, 5, or 6.*

type i

1. $e \in E(G)$.
2. $w \in B$ and e corresponds to a path $vx_e w$ in G with $x_e \in S_1$.
3. $w \in B$ and e corresponds to a path $vy_e x_e w$ in G with $x_e \in S_1$ and $d(y_e) = 2$.
4. $w \in S$ and e corresponds to a path $vy_e w$ in G with $d(y_e) = 2$.
5. e corresponds to a path or cycle $vx_e x'_e w$ in G with $x_e, x'_e \in S_1$.
6. e corresponds to a path $vx_e y_e x'_e w$ in G with $x_e, x'_e \in S_1$ and $d(y_e) = 2$.

Proof. Due to the construction of G' , each edge e in G' between v and w comes from a path (or cycle) P_e in G between v and w . In particular, every internal vertex of P_e is either a 2-vertex in $S \setminus N[B]$ or a vertex of S_1 which is preceded or followed in P_e by a big vertex. This implies that each internal vertex of P_e is small, and that the only vertices of P_e that can be big are v and w .

By Lemma 3.4, no two consecutive vertices of P_e are suppressed. This implies that P_e has length at most four.

- If P_e has length one, then e has type 1.
- If P_e has length two, then we have $v \neq w$ since G is simple. Denote by x the middle vertex of P_e . We must have either $v, w \in S$ and $d_G(x) = 2$ (case 4), or $v \in B$, $w \in S$ and $x \in S_1$ (case 2).
- If P_e has length three, then at least one of v, w must be in B and its neighbor in P_e must be in S_1 . If both v and w lie in B , then we are in case 5; otherwise, we have $v \neq w$ and we are in case 3.
- Finally, if P_e has length four, then we have $v \neq w$ since G is C_4 -free. Moreover, they both have to be big and their neighbors in P_e (say x_e, x'_e) lie in S_1 . The other vertex y_e of P_e must have degree two, so we are in case 6.

Observe in particular that if v is small, then cases 5 and 6 cannot occur. Moreover, if v and w are big, then only cases 1, 5, and 6 can occur. Finally, every loop of G' has type 5. \square

In what follows, when referring to an edge e with type i , we use x_e, x'_e , and y_e as defined in the corresponding part of Lemma 3.10. This lemma implies the following facts about the structure of regions in G .

Corollary 3.11. *Let R be a region of G . Then $V(R)$ is the disjoint union of three sets B_1, B_2, D such that $B_i \subset N(b_i)$ for some $b_1, b_2 \in B$, and D is an independent set of 2-vertices, each with a neighbor in each of B_1 and B_2 .*

Proof. Let R be a region of G . By definition, there exists $b_1, b_2 \in B$ on the boundary of every face of R in G'' . Therefore, in G' , the edges appearing in R are either loops on b_1 or b_2 or edges between b_1 and b_2 .

Note that $V(R)$ is the set of all vertices of G that disappear when we construct the edges of R in G' . For each $i \in \{1, 2\}$, define B_i as the set of vertices v of G such that vb_i is contracted when constructing an edge of R in G' . We also define D as the set of vertices in G that are suppressed when constructing an edge of R in G' . By definition, we have $B_i \subset N(b_i)$

By Lemma 3.10, since $b_1, b_2 \in B$, each edge e between b_1 and b_2 in G' has type 1, 5, or 6, and each loop around b_1, b_2 has type 5. This ensures that $V(R) = B_1 \cup B_2 \cup D$ and that D contains only vertices of degree 2 in G . Using again Lemma 3.10, this implies that D is an independent set.

It remains to show that these sets are pairwise disjoint. Assume that there is $x \in B_1 \cap B_2$. Now xb_1 and xb_2 are both contracted when constructing G' . This requires that $x \in S_1$. Since b_1 and b_2 are both big, we must have $b_1 = b_2$, a contradiction. Further, since $b_1 \in B$, no neighbor of b_1 is suppressed during the construction of G' . Since $B_1 \subset N(b_1)$, we thus have $D \cap B_1 = \emptyset$. By symmetry, we also have $D \cap B_2 = \emptyset$. \square

In the following, given a region R , we use the notation of Corollary 3.11.

3.4 Proof of Proposition 3.8: G has Large Regions

Our goal in this subsection is to find a large region in G . To this end, we look for a large set of consecutive faces of length 2 in G' . We first recall a result from [3] (Lemma 3.6 in that paper) allowing us to find a vertex in G' with few neighbors in G''' .

Lemma 3.12 ([3]). *There exists $b_1 \in B$ such that $d_{G'''}(b_1) \leq 40$ and $d_{G'''}[B](b_1) \leq 10$.*

We note that the general context of [3] is planar graphs with girth at least 5. However, the proof of Lemma 3.12 uses only that G has no 4-cycles.

Our goal is to apply a pigeonhole-like argument to find a large number of consecutive edges between two vertices in G'' . To this end, we first need to control the degrees of vertices in G'' . We begin with a definition. The *half-edges* of G' are the elements of the multiset of pairs (u, e) where e is an edge incident to u . Note that when e is a loop around u , there are still two half-edges (u, e) . Observe also that since we fixed a plane embedding of G , there is a natural cyclic ordering of the half-edges around each fixed vertex u .

half-edges

Lemma 3.13. *If e is a loop around a vertex v in G' , then one of the half-loops induced by e must be followed or preceded by a half-edge (v, vw) with $v \neq w$.*

Proof. By Lemma 3.10, every loop has type 5. So let x_e and x'_e denote the vertices in G that merged into v to form e in G' . By Lemma 3.5, either $d(x_e) > 2$ or $d(x'_e) > 2$; by symmetry, assume $d(x_e) > 2$. Among all neighbors of x_e in G , other than x'_e and v , choose w to be one that immediately precedes or follows x'_e .

If w is not suppressed in G' , then the half-edge (v, vw) precedes or follows (v, e) or (v, e') . Note that $vw \notin E(G)$ since otherwise $vwx_ex'_e$ is a 4-cycle in G . Thus we have $v \neq w$ in G' and the lemma is true. So assume that w is suppressed. Now w has degree 2 in G . Let x be the neighbor of w other than x_e . Since x_e is small, Lemma 3.4 ensures

that x has degree at least 3 in G ; hence, it is not suppressed in G' . Therefore, the half-edge (v, vx) precedes or follows (v, e) or (v, e') . Again, $vx \notin E(G)$ since otherwise $vxwx_e$ is a 4-cycle in G . Thus $x \neq v$ in G' and the lemma is true. \square

Lemma 3.13 implies the following relationship between degrees of vertices in G'' and in G' .

Corollary 3.14. *Every $v \in V(G')$ satisfies $d_{G''}(v) \geq \frac{d_{G'}(v)}{5}$.*

Proof. Suppose $v \in V(G')$ and consider the half-edges around v in G' . By definition, there are $d_{G'}(v)$ half-edges around v and $d_{G''}(v)$ of them are not half-loops. So it suffices to prove that the number of half-loops around v is at most four times the number of the other half-edges, i.e., at most $4d_{G''}(v)$.

Suppose $w \in N_{G'}(v)$. Consider the two half-edges (v, e) and (v, f) such that (v, e) , (v, vw) and (v, f) are consecutive around v . Let $F(w)$ be the maximum subset of $\{(v, e), (v, f)\}$ containing only half-loops. Lemma 3.13 ensures that, for every loop, one of its half-loops appears in $F(w)$ for some $w \in N_{G'}(v)$. Therefore, the number of half-loops around v is at most

$$2 \left| \bigcup_{w \in N_{G'}(v)} F(w) \right| \leq 4|N_{G'}(v)| = 4d_{G''}(v).$$

This concludes the proof, since

$$d_{G'}(v) \leq d_{G''}(v) + 4d_{G''}(v) = 5d_{G''}(v). \quad \square$$

Consider the vertex b_1 obtained by Lemma 3.12. By Corollary 3.14, we have

$$d_{G''}(b_1) \geq \frac{d_{G'}(b_1)}{5} \geq \frac{d_G(b_1)}{5} \geq \frac{\sqrt{k}}{5}.$$

Using a pigeonhole argument, we will see that b_1 has some neighbor b_2 such that at least $\frac{\sqrt{k}}{5 \times 40}$ consecutive edges incident to b_1 end at b_2 . Note that Proposition 3.8 almost follows from this result (with $\frac{\sqrt{k}}{50}$ replaced by $\frac{\sqrt{k}}{200}$). We only need to refine this argument to show how to force $b_2 \in B$, i.e., $b_2 \notin S'$, where $S' = V(G') \setminus B$. To this end, we show that small vertices are incident to few consecutive edges in G'' .

Lemma 3.15. *If $v \in B$ and $w \in S'$, then (v, w) is on the boundary of at most 8 consecutive faces of length 2 in G'' .*

Proof. Pick $v \in B$ such that there is an edge $vw \in E(G')$, with $w \in S'$. We consider each possible type of edge in G' between v and w . The type 3 edges are a special case, which we postpone to the end. Since G is simple, at most one edge vw of G' has type 1. Similarly, if G' has two edges e_1 and e_2 of type 2, then $x_{e_1} \neq x_{e_2}$. Thus $vx_{e_1}wx_{e_2}$ is a 4-cycle in G , a contradiction. So G' has at most one edge of type 2. Since $v \in B$ and $w \in S'$, G' has no edge of type 4, 5, or 6.

Only type 3 edges remain. We assume such an edge exists, since otherwise the lemma holds. Note that G' has no edge of type 4 (since $v \in B$), nor of type 1 (since G has no 4-cycle), nor of type 5 or 6 (since $w \in S'$). So G' has at most one edge f not of type 3, and f , if it exists, has type 2. Thus, edge f separates two blocks of consecutive

type 3 edges. To prove the lemma, it suffices to prove that each such block has size at most four.

Assume that e_1, \dots, e_5 are edges of type 3 that are consecutive in G'' . We now prove that the hypotheses of Lemma 3.7 are satisfied by the subgraph of G induced by the vertices inside the cycle $vx_{e_1}y_{e_1}wy_{e_5}x_{e_5}$. Since each edge e_i has type 3, the first hypothesis holds.

To prove the second hypothesis holds, assume that some vertex x is not adjacent to v , but x lies inside some cycle $C = vx_{e_i}y_{e_i}wy_{e_{i+1}}x_{e_{i+1}}$. Note that x is not a neighbor of y_{e_i} or $y_{e_{i+1}}$, since they both have degree 2; nor of w since e_i and e_{i+1} are consecutive edges in G'' . Note that e_i and e_{i+1} bound a face of length 2 in G'' so every vertex inside the cycle C disappears when we construct G' . Thus, all these vertices are small, and either lie in S_1 or lie in $S \setminus N[B]$ and have degree 2 in G . Hence, v is the only big vertex inside or on C and $xv \notin E(G)$; so $x \notin \cup_{i \geq 1} S_i$.

Since $x \notin S_1$, x has degree 2 and its two neighbors, say y and z , lie in S . Applying Lemma 3.4 to edges xy and xz , we get that $y, z \in N[B]$. This implies that both y and z are neighbors of v , so $xyvz$ is a 4-cycle in G , a contradiction. Therefore, no such x exists.

Now Lemma 3.7 yields a contradiction, since G cannot contain this configuration. \square

We can now finish the proof of Proposition 3.8.

Proof of Proposition 3.8. Let b_1 be a vertex in G''' guaranteed by Lemma 3.12. For each small neighbor v of b_1 in G''' and edge vb_1 , Lemma 3.15 ensures that in G'' edge vb_1 corresponds to at most 9 edges between b_1 and v . Since $d_{G'''}(b_1) \leq 40$, the number of such edges is at most $9 \times 40 = 360$. However, by Corollary 3.14, we have $d_{G''}(b_1) \geq \frac{d_{G'''}(b_1)}{5} \geq \frac{\sqrt{k}}{5}$. Thus, there must exist a big neighbor b_2 of b_1 in G'' such that there are at least $\frac{\frac{\sqrt{k}}{5} - 360}{d_{G'''}(b_1)} \geq \frac{\sqrt{k}}{50} - 36$ consecutive edges b_1b_2 in G'' . By definition, these edges form a region of size $\frac{\sqrt{k}}{50} - 37$ in G . \square

3.5 Proof of Proposition 3.9: Large Regions are Reducible

In this section, we show that G cannot contain arbitrarily large regions, i.e., for r large enough every r -region is reducible. Note that the square of such r -regions consists of two cliques, with some edges between them. Following the terminology of Corollary 3.11, we denote the vertices of these cliques by B_1 and B_2 . As before, D denotes a set of independent 2-vertices, each with one neighbor in B_1 and one neighbor in B_2 . We begin by proving that there are only few edges between B_1 and B_2 .

Lemma 3.16. *Let R be an r -region of G . Each $w \in B_1 \cup B_2$ has at most one neighbor in B_1 , at most one in B_2 , and at most eight in D .*

Proof. Suppose $w \in B_1 \cup B_2$. If w has two neighbors x and y in B_i , then wxy is a 4-cycle in G , a contradiction. So we assume w has at most one neighbor in each of B_1 and B_2 . In what follows, we assume by symmetry that $w \in B_1$.

Suppose that w has 5 consecutive neighbors x_1, \dots, x_5 , all in D , and denote by y_i the common neighbor of x_i and b_2 . By Lemma 3.7, there is a vertex z inside some cycle $wx_iy_ib_2y_{i+1}x_{i+1}$ that is not adjacent to b_2 . Since R is an r -region, z disappears when we construct G' . Since $z \notin N_G(b_2)$, vertex z must be a 2-vertex. By Lemma 3.4, each neighbor of z is adjacent to b_2 . So G contains a 4-cycle, a contradiction. Thus, w has at most 4 consecutive neighbors in D .

Consider an edge wx between these blocks of consecutive neighbors in D where $x \in V(R) \setminus D$. Then x cannot lie in B_1 , otherwise b_1wx is a triangle not containing b_2 nor any vertex in B_2 . By planarity, there cannot be vertices of D inside and outside of this triangle. Therefore $x \in B_2$.

Since G has no 4-cycle, at most one such neighbor x exists, so w has at most two blocks of consecutive neighbors in D . This proves the final assertion. \square

We note that we can prove Theorem 3.1 more simply (and with a better bound on Δ) if we only want the result for list-coloring. In fact, we do this in Section 3.6, where we prove it for correspondence coloring. However, to prove this same bound for Alon–Tarsi number, as we do below, seems to require using the Kernel Lemma (Lemma 3.17).

Proving that G does not contain large regions amounts to proving that r -regions of G are square L' -colorable for a suitable assignment L' . To prove this new assertion, we use an auxiliary result about choosability, due to Bondy, Boppana, and Siegel (see Remark 2.4 in [1]). This result applies to kernel perfect digraphs. We briefly recall the definition here. A *kernel* K in a digraph D is a subset of $V(D)$ such that every vertex v of D satisfies: $v \in K$ if and only if $N^+(v) \cap K = \emptyset$. A digraph is *kernel perfect* if each of its induced subgraphs has a kernel.

kernel
kernel perfect

Lemma 3.17. *Let D be a kernel perfect digraph D with underlying graph H . If L is a list assignment for $V(H)$ such that for all $v \in V(H)$, $|L(v)| \geq d^+(v) + 1$, then H is L -colorable.*

We use this lemma to reduce the problem of square L -coloring an r -region to finding a kernel perfect orientation. We apply this method to prove the following generic result about choosability of graphs covered by two cliques with few edges between them. (Our next lemma is analogous to Lemma 3.13 in [3]. One major reason that our bounds below on $|B_i|$ and $|T_i|$ are so much larger is that here we apply the lemma to a graph G that can contain triangles.)

Lemma 3.18. *Let H be a graph covered by two disjoint cliques, B_1 and B_2 . Let L be a list assignment for $V(H)$ and suppose $T_i \subset B_i$ for each $i \in \{1, 2\}$. Now H is L -colorable if the following five conditions hold.*

1. $|B_1| \geq 52811$ and $|B_2| \geq 52811$.
2. $|T_1| \leq 4400$ and $|T_2| \leq 4400$.
3. For each $v \in B_i$, $|N(v) \cap B_{3-i}| \leq 11$.
4. For each $v \in T_i$, $|L(v)| \geq |B_i| - 44$.
5. For each $v \in B_i \setminus T_i$, $|L(v)| \geq |B_i|$.

Proof. To prove this result we construct an orientation D of H such that D satisfies the hypotheses of Lemma 3.17. We first show that we can order the vertices $x_1, \dots, x_{|B_1|}$ and $y_1, \dots, y_{|B_2|}$ of B_1 and B_2 such that $T_1 = \{x_1, \dots, x_{|T_1|}\}$, $T_2 = \{y_1, \dots, y_{|T_2|}\}$ and every path beginning and ending in $\{x_{|B_1|-10}, \dots, x_{|B_1|}, y_{|B_2|-10}, \dots, y_{|B_2|}\}$ that alternates between B_1 and B_2 has length at least 4. Note that a single edge may be an alternating path, so we require that no edge joins x_i and y_j whenever $i \geq |B_1| - 10$ and $j \geq |B_2| - 10$.

Definition of the Orderings

We now construct the vertex orderings in the previous paragraph. Their only non-trivial property is the absence of short alternating paths between the final 11 vertices in B_1 and those in B_2 . So, our goal is to construct $Z_1 \subset B_1$ and $Z_2 \subset B_2$ with $|Z_1| = |Z_2| = 11$ such that no alternating path of length at most 3 begins in Z_1 and ends in Z_2 . To this end, we first define Z_2 , then count the number of vertices in B_1 reachable from Z_2 with such an alternating path.

If there exists $v \in B_1 \setminus N(T_2)$ with 11 neighbors in B_2 , then we take $Z_2 = N_H(v) \cap B_2$. If no such vertex exists, then we swap the roles of B_1 and B_2 , take Z_2 as any subset of $B_2 \setminus (T_2 \cup N(T_1))$ of size 11 (this is always possible since $|B_2| \geq 52811 \geq |T_2| + 11|T_1| + 11$), and let v be any vertex of B_1 . Since every element of Z_2 has at most 10 neighbors in $B_1 \setminus \{v\}$, we have $|N_{B_1}(Z_2) \setminus \{v\}| \leq 11 \times 10 = 110$. Moreover, each vertex in $N_{B_1}(Z_2) \setminus \{v\}$ has at most 11 neighbors in B_1 (one of them being in Z_2). Since the only neighbors of v in B_2 are in Z_2 , we obtain

$$|N_{B_2}(N_{B_1}(Z_2)) \setminus Z_2| \leq 11 \times 10^2 = 1100.$$

By the same argument, the set of vertices of B_1 reachable from Z_2 with an alternating path of length exactly 3 has size

$$|N_{B_1}(N_{B_2}(N_{B_1}(Z_2)) \setminus Z_2)| \leq 1100 \times 10 = 11000.$$

So the number of vertices of B_1 that are excluded from appearing in Z_1 , because of paths to Z_2 , is at most

$$|N_{B_1}(N_{B_2}(N_{B_1}(Z_2)) \setminus Z)| + |N_{B_1}(Z_2) \setminus \{v\}| + |\{v\}| = 11000 + 110 + 1 = 11111.$$

Further, we must also remove vertices of T_1 . Thus, we can choose Z_1 as desired, since $|B_1| - |T_1| - 11111 \geq 11$.

Definition of the Orientation

For each edge with both endpoints in the same clique, direct it toward the vertex of lower index. For every other edge, direct it in both directions, unless one of its endpoints is among the last 11 vertices of B_1 or B_2 . In this case, direct the edge toward this endpoint.

The Orientation is Kernel-perfect

Let $A \subseteq V(H)$, with $A \neq \emptyset$. We look for a kernel of A . Let x_p (resp. y_q) denote the vertex with smallest index in $A \cap B_1$ (resp. $A \cap B_2$), if it exists. If $A \cap B_1 = \emptyset$, then $\{y_q\}$ is a kernel. Similarly, if $A \cap B_2 = \emptyset$, then $\{x_p\}$ is a kernel. So we assume that both x_p and y_q are well-defined. We can also assume that $x_p y_q \in E(H)$, since otherwise $\{x_p, y_q\}$ is a kernel.

A, x_p, y_q

Let x_r (resp. y_s) denote the vertex with smallest index in $A \cap B_1$ (resp. $A \cap B_2$) that is not a neighbor of y_q (resp. x_p).

x_r, y_s

We now prove that at least one of $\{x_p\}$, $\{x_p, y_s\}$, $\{y_q\}$ and $\{x_r, y_q\}$ is a kernel. Assume the contrary. Since $\{x_p, y_s\}$ is not a kernel, there exists y_j such that $q \leq j < s$ and either there is no edge $x_p y_j$ or it is directed only towards y_j . Due to the choice of s , this edge is present in H and is thus directed only one way. (If y_s is not well-defined, i.e. if x_p is adjacent to every vertex in $A \cap B_2$, we can obtain the same result using that $\{x_p\}$ is not a kernel.)

Similarly, using that $\{x_r, y_q\}$ is not a kernel (or only $\{y_q\}$ if y_q is adjacent to every vertex in $A \cap B_1$), we have an edge $x_i y_q$ directed only towards x_i .

Since $x_i y_q$ and $x_p y_j$ are directed towards x_i and y_j , this ensures that x_i and y_j are both among the final 11 vertices of B_1 and B_2 . However, this is impossible, since $x_i y_q x_p y_j$ would be a path of length 3 that alternates between B_1 and B_2 and begin and ends in the final 11 vertices of B_1 and B_2 . Thus, either $\{x_p, y_s\}$, $\{x_p\}$, $\{x_r, y_q\}$ or $\{y_q\}$ is a kernel of A . So the orientation is kernel-perfect.

The Orientation has Small Out-degrees

We now prove that $|L(v)| \geq d^+(v) + 1$ for every $v \in V(H)$. By symmetry, it suffices to prove this for all $v \in B_1$, i.e., $v = x_i$ whenever $i \in \{1, \dots, |B_1|\}$. If $i \leq |T_1|$, i.e., $v \in T_1$, then v has at most $|T_1| - 1 \leq 4399$ out-neighbors in B_1 and at most 11 out-neighbors in B_2 . So $d^+(v) + 1 \leq 4411 \leq |B_1| - 44 \leq |L(v)|$. If $|T_1| < i \leq |B_1| - 11$, then v has at most $|B_1| - 12$ out-neighbors in B_1 and at most 11 in B_2 . So $d^+(v) + 1 \leq |B_1| \leq |L(v)|$. If $i > |B_1| - 11$, then every out-neighbor of v is in B_1 , so $d^+(v) + 1 \leq |B_1| \leq |L(v)|$. \square

We now use this lemma to prove Proposition 3.9, i.e., that large regions are reducible for square choosability.

Proof of Proposition 3.9. We use proof by contradiction. Assume that G has an r -region R with $r \geq 475353$. Let v_1 and v_2 be adjacent vertices of R such that any vertex at distance 2 in G from $\{v_1, v_2\}$ lies in $\{b_1, b_2\} \cup V(R) \cup N(b_1) \cup N(b_2)$. To see that such vertices exist, pick $v_1 \in B_1$ such that each face containing v_1 is in R , and let v_2 be a neighbor of v_1 in $B_2 \cup D$.

Let T denote the set of vertices in $B_1 \cup B_2$ that appear on a face of G not in R . Note that $|T| \leq 4$; this is because each vertex of T must lie on the first or last edge of the r -region in G' , and each of these edges has exactly one vertex in each of B_1 and B_2 . Let $T^{(1)} = N(T) \cap V(R)$, $T^{(2)} = N(T^{(1)}) \cap V(R)$ and $T^{(3)} = N(T^{(2)}) \cap V(R)$, so that for $1 \leq i \leq 3$, $T \cup \dots \cup T^{(i)}$ is the set of vertices of $V(R)$ at distance at most i from T (with the distance taken in $V(R)$). By Lemma 3.16, each vertex of T has at most 10 neighbors in $V(R)$, so $|T^{(1)}| \leq 40$, $|T^{(2)}| \leq 400$ and $|T^{(3)}| \leq 4000$.

T

$T^{(1)}, T^{(2)}$
 $T^{(3)}$

By minimality, $(G - v_1v_2)^2$ has an L -coloring φ . Let $B'_i = B_i \setminus N[T]$. We uncolor the vertices of $B'_1 \cup B'_2 \cup D$. We also define T_i as the set of vertices of B'_i with some colored neighbor from $V(R)$ in G^2 , i.e., $T_i = B'_i \cap (T^{(2)} \cup T^{(3)})$. Finally, let $H = G^2[B'_1 \cup B'_2]$. Note that B'_1 and B'_2 are cliques in H . Moreover, they are disjoint since $B'_1 \cap B'_2 \subset B_1 \cap B_2 = \emptyset$. φ, B'_i
 T_i, H

Our goal is now to apply Lemma 3.18 to L' -color H , where L' is the list assignment formed from L by removing all colors already used on vertices at distance at most 2:

$$L'(v) = L(v) \setminus \{\varphi(w), w \in N^2(v) \setminus (V(H) \cup D)\}. \quad L'(v)$$

We prove that the hypotheses of Lemma 3.18 are satisfied.

Suppose $v \in B'_1$. Now $|N^2(v) \cap B'_2| = |N(v) \cap B'_2| + \sum_{w \in N(v)} |N(w) \cap B'_2|$. By Lemma 3.16, for each $w \in V(R)$, $|N(w) \cap B'_2| \leq 1$. Moreover, if $w \notin V(R)$, then $|N(w) \cap B'_2| = 0$, unless $w = b_2$. Since $b_2 \notin N(v)$, we get

$$|N^2(v) \cap B'_2| \leq 1 + |N(v) \cap V(R)| \leq 11.$$

Suppose $v \in B'_1 \setminus T_1$. By definition, v is distance at least four from T (in $V(R)$), hence at distance at least three (in $V(R)$) from $N[T]$, the set of colored vertices of $V(R)$. So the only colored neighbors of v in G^2 are in $\{b_1, b_2\} \cup (N(b_1) \setminus B'_1)$. Hence, we have

$$|L'(v)| \geq k + 2 - (2 + k - |B'_1|) = |B'_1|.$$

Suppose $v \in T_1$. By construction, its colored neighbors in G^2 are in $\{b_1, b_2\} \cup (N(b_1) \setminus B'_1) \cup T \cup T^{(1)}$. Since $|T| + |T^{(1)}| \leq 44$, we have $|L'(v)| \geq |B'_1| - 44$.

We already saw that $|T_1| \leq |T^{(2)} \cup T^{(3)}| \leq 400 + 4000 = 4400$. There are $r + 1$ edges in the region R (in G'). Every such edge (except b_1b_2 if it exists) corresponds to a path containing a vertex in B_1 . By Lemma 3.16, each vertex in B_1 accounts for at most nine of them. Therefore, $|B_1| \geq \frac{r}{9}$. Observe also that $|N[T] \cap B_1| \leq 6$ since $|T \cap B_1| = 2$ and, by Lemma 3.16, every vertex of $B_1 \cup B_2$ has at most one neighbor in each of B_1 and B_2 . We thus obtain:

$$|B'_1| \geq |B_1| - |N[T] \cap B_1| \geq \frac{r}{9} - 6 \geq 52811.$$

We can thus apply Lemma 3.18 to find an L' -coloring of H .

It remains to color the vertices in D . Note that each has at most $2\sqrt{k}$ neighbors and $k + 2$ colors. So we can greedily color the vertices in D . \square

This completes the proof of Theorem 3.1.

3.6 Extension to correspondence coloring

In this section, we prove the following extension of Theorem 3.1 to correspondence coloring. (Recall the definition of correspondence coloring from the end of Section 2.1.)

Theorem 3.19. *There exists Δ_0 such that if G is a plane graph with no 4-cycles and with $\Delta(G) \geq \Delta_0$, then $\chi_{\text{corr}}(G^2) \leq \Delta + 2$.*

Let $\Delta_0 = 2642900^2 = 6984920410000$, and fix $k \geq \Delta_0$. We prove Theorem 3.19 by contradiction. Suppose the theorem is false; let G be a counterexample minimizing $|V(G)| + |E(G)|$, and let C be a $(k+2)$ -correspondence assignment for G^2 such that G^2 has no C -coloring. So C assigns, to each pair of vertices (v, w) adjacent in G^2 , a partial matching C_{vw} between $\{v\} \times \{1, \dots, k+2\}$ and $\{w\} \times \{1, \dots, k+2\}$.

Δ_0, k

We claim that Lemmas 3.2 through 3.16 still hold for G in this new setting, since in proving each lemma we color vertices using only that they have more available colors than colored neighbors. So Proposition 3.8 also still holds. It thus suffices to prove the following generalization of Proposition 3.9 for G .

Proposition 3.20. *Every r -region of G satisfies $r \leq 52821$.*

Assuming this proposition holds, we can conclude. Indeed, Propositions 3.8 and 3.20 imply that $\frac{\sqrt{k}}{50} - 37 < 52821$, i.e., that $k < 2642900^2 = 6984920410000 = \Delta_0$, a contradiction.

It thus remains to prove that large regions are reducible, by generalizing Lemma 3.18. The argument using kernel-perfect orientations is no longer valid, since Lemma 3.17 does not extend to correspondence coloring.

Lemma 3.21. *Let H be a graph covered by two disjoint cliques, B_1 and B_2 , each of size n . Suppose there exist $T_1 \subset B_1$ and $T_2 \subset B_2$, and a function f satisfying the four properties below. If $n \geq 5863$, then every f -correspondence assignment C admits a C -coloring.*

1. For each $v \in (B_1 \setminus T_1) \cup (B_2 \setminus T_2)$, we have $f(v) \geq n$.
2. For each $v \in T_1 \cup T_2$, we have $f(v) \geq n - 44$.
3. $|T_1| \leq 4400$ and $|T_2| \leq 4400$.
4. $\Delta(H) - n + 1 \leq 11$.

Proof. Let A be a subset of $B_1 \setminus T_1$ with $|A| = \Delta(H) + 1 - n$. Since each vertex $v \in (B_1 \setminus T_1) \cup (B_2 \setminus T_2)$ has $f(v) \geq n$ and $\Delta(H) - |A| = n - 1$, it is easy to greedily C -color all vertices of $H - A$. For example, greedily color all vertices of T_2 , followed by those of $B_2 \setminus T_2$, followed by those of T_1 , followed by those of $B_1 \setminus (T_1 \cup A)$. This greedy coloring is possible because at the time we color each vertex it has more available colors than colored neighbors.

A

We generally follow this approach. However, we modify it so that after we color $H - A$ each vertex in A still has $|A|$ available colors, and we can extend the coloring to A . To do this, for each vertex $v \in A$ we will repeatedly “save a color”, before greedily coloring the other vertices. To accomplish this we pick vertices $w \in N(v) \cap B_2$ and $x \in B_1 \setminus N(w)$. Now we color w and x with some colors α and β (possibly with $\alpha = \beta$) such that α and β forbid the same color on v . For each $v \in A$, we must save a color $|N(v) \cap B_2|$ times. After doing so, we color the remaining vertices greedily (as in the previous paragraph), ending with the vertices of A . The only change is that we must ensure that each of the final 11 vertices we color in B_2 has no colored neighbor in B_1 . In the process of saving colors for vertices in A , we color at most 11^2 vertices in B_1 . Each of these forbids at most 11 vertices in B_2 from appearing among the final 11 in B_2 , for a total of at most 11^3 vertices in B_2 forbidden. Similarly, we color at most 11^2

v
 w
 x, α, β

vertices in B_2 , and these are obviously forbidden from appearing among the final 11 vertices in B_2 . Thus, we can choose the desired 11 final vertices in B_2 (after saving colors for the vertices in A), since $|B_2| \geq |T_2| + 11^3 + 11^2 + 11$.

Note that, while saving colors for some vertex $v \in A$, we color all neighbors of v in B_2 . As a result, we need that no two vertices in A have a common neighbor in B_2 . Each vertex $v \in A$ has at most 11 neighbors in B_2 , and each of these neighbors has at most 10 other neighbors in B_1 . Thus, each $v \in A$ forbids at most $11(10)$ other vertices from A . So, to pick the desired A , we need $|B_1| > |T_1| + 10(110 + 1)$.

Now, for each $v \in A$, we repeat the following $|N(v) \cap B_2|$ times. Choose uncolored vertices $w \in N(v) \cap B_2$ and $x \in B_1 \setminus N(w)$. Note that if $N(v) \subset B_1$, there is nothing to do at all, hence we may assume that the vertex w exists. Let $g(v)$, $g(w)$, and $g(x)$ denote the number of remaining available colors for v , w , and x .

$g(v), g(w)$
 $g(x)$

Without loss of generality, we assume that the bounds of Hypotheses 1. and 2. are tight, so that $f(y) = n - 44$ for all $y \in T_1 \cup T_2$, and $f(y) = n$ otherwise. Since $A \cap T_1 = \emptyset$, we have $f(v) = n \geq f(w)$, hence we may assume that C_{vw} saturates $\{w\} \times \{1, \dots, f(w)\}$ (otherwise, add arbitrary edges until this is the case). Thus, each color available for w forbids a color for v ; similarly for colors available for x . By Pigeonhole, if $g(w) + g(x) > n$, then there exist colors α and β , available for w and x respectively, that both forbid the same color on v . Suppose that thus far we have saved a total of i colors for vertices in A . Therefore, the i colored vertices of B_2 forbid i colors for w , and its neighbors in B_1 forbid at most 11 colors, so that we have $g(w) \geq f(w) - i - 11 \geq n - i - 11 \geq n - 131$ and, similarly, $g(x) \geq n - 131$. We can assume that $g(v) \leq f(v) \leq n$. And clearly $2(n - 131) > n$. Thus, the desired colors α and β exist. \square

It is worth noting that the Δ_0 given by our proof of Theorem 3.19, namely 2642900^2 , is much smaller than that arising from our proof of Theorem 3.1, namely 23769500^2 . By adapting the statement and proof of Lemma 3.21, we can extend the main result in [3] to correspondence coloring (while also modestly decreasing the Δ_0 arising from that proof).

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